ENDTERM COMPLEX FUNCTIONS
JUNE 27 2012, 9:00-12:00

• Put your name and student number on every sheet you hand in.
• When you use a theorem, show that the conditions are met.

Exercise 1 (7 pt) Compute
\[ \sum_{n=0}^{\infty} \frac{\sin(nt)}{n!} \quad (t \in \mathbb{R}) \]

Hint: Rewrite the series using the exponential function.

Exercise 2 (20 pt) Prove that the following integrals converge and evaluate them.

a. (10 pt) \[ \int_0^{\infty} \frac{1}{(x^2 + i)^2} \, dx \]
b. (10 pt) \[ \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx \]

Exercise 3 (10 pt) Let \( f \) be an entire function satisfying \( |f(-z)| < |f(z)| \) for all \( z \) in the upper halfplane (\( \text{Im}(z) > 0 \)).

a. (7 pt) Prove that \( g(z) = f(z) + f(-z) \) can only have real roots.
b. (3 pt) Prove that \( z \sin(z) = \cos(z) \) only has real solutions.

Exercise 4 (8 pt) Is there an analytic isomorphism between the open unit disc \( D \) and \( \mathbb{C} \setminus \{a\} \) with \( a \in \mathbb{C} \)?

Bonus exercise (15 pt) Let \( f : \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x = 1\} \to \mathbb{C} \) be the sum of \( (\log z)^{-2} \) along all the branches of the logarithm, i.e.

\[ f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(\log(z) + 2\pi in)^2} \]

a. (5 pt) Prove that \( f \) is meromorphic on \( \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\} \).
b. (5 pt) Prove that \( f \) can be analytically continued to \( \mathbb{C} \setminus \{1\} \).
c. (5 pt) Prove this analytic continuation is a rational function.