ENDTERM COMPLEX FUNCTIONS

JULY 01 2014, 8:30-11:30

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Exercise 1 (*10 pt*): Consider a transformation of the complex plane

$$z \mapsto a\bar{z} + b,$$

where $a, b \in \mathbb{C}$ with $|a| = 1$. Prove that this transformation has a straight line composed of fixed points if and only if

$$-a\bar{b} = b.$$

Exercise 2 (*10 pt*): Let $m > 0$ be integer. Find the convergence radius of the following series

$$\sum_{n=0}^{\infty} (a_1^n + a_2^n + \cdots + a_m^n) z^n,$$

where $a_j \in \mathbb{C}$ with $|a_j| = 1$ for $j = 1, 2, \ldots, m$.

Exercise 3 (*15 pt*): Let $\Omega \subset \mathbb{C}$ be open and bounded. We define the *Cauchy-Riemann operator* by

$$\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Prove that the boundary value problem

$$\begin{cases} \partial_{\bar{z}} u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

for given continuous functions $f : \Omega \to \mathbb{C}$ and $g : \partial\Omega \to \mathbb{C}$ has at most one solution $u : \overline{\Omega} \to \mathbb{C}$ that is continuous in $\overline{\Omega}$.

Turn the page!

Exercise 4 (*20 pt*): Let $f(z) = z^6 - 5z^4 + 10$.

a. (*15 pt*) Prove that f has

(i) no zeroes with $|z| < 1$;
(ii) 4 zeroes with $|z| < 2$;
(iii) 6 zeroes with $|z| < 3$.

b. (*5 pt*) For cases (ii) and (iii), show that all zeroes are different.

Exercise 5 (*25 pt*): Prove that the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

converges and compute it.

Hint: As one possibility is to consider the integral of the function $\dfrac{e^{iz}}{z^3}$ over an appropriate closed path and prove that

$$\int_{\rho}^{\infty} \frac{\sin x}{x^3} dx = \frac{1}{\rho} - \frac{\pi}{4} + O(\rho), \quad \rho \to 0,$$

from which one can deduce

$$\int_{\rho}^{\infty} \frac{\sin 3x}{x^3} dx = \frac{3}{\rho} - \frac{9\pi}{4} + O(\rho), \quad \rho \to 0.$$

Bonus Exercise (*10 pt*): Let a function $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose moreover that f is analytic for both $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, z < 0$. Prove that f is analytic on $\mathbb{C}$.