Exercise 1. (30 pt) In this exercise, we will compute the total derivative of the inversion mapping \( G : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \) defined by
\[
G(x) = \frac{1}{\|x\|^2} x,
\]
where \( \|x\| \) is the standard norm in \( \mathbb{R}^n \), i.e. \( \|x\|^2 = \langle x, x \rangle = x^T x \).

(a) (5 pt) Describe the action of the mapping (1) geometrically.

\( G \) inverts the distance of points to the origin. It preserves all radial rays and interchanges the sphere of radius \( r \) centred at the origin with that of radius \( \frac{1}{r} \).

(b) (10 pt) Let \( U \subset \mathbb{R}^n \) be open and let \( f : U \rightarrow \mathbb{R} \) and \( G : U \rightarrow \mathbb{R}^n \) be two differentiable mappings. Define \( fG : U \rightarrow \mathbb{R}^n \) via \((fG)(x) = f(x)G(x), \ x \in U \). Prove that \( fG \) is differentiable and
\[
D(fG)(x) = f(x)DG(x) + G(x)Df(x), \quad x \in U.
\]

This can be done in several ways:

1. Let \( x \in U \), then by Hadamard’s lemma there exist continuous functions \( \phi : U \rightarrow \text{Lin}(\mathbb{R}^n, \mathbb{R}) \) and \( \Gamma : U \rightarrow \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \) such that
\[
f(y) = f(x) + \phi(y)(y-x) \quad \text{and} \quad G(y) = G(x) + \Gamma(y)(y-x)
\]
for all \( y \in U \). Moreover, \( \phi(x) = Df(x) \) and \( \Gamma(x) = DG(x) \).

Consequently, we find that
\[
(fG)(y) = f(y)(G(x) + \Gamma(y)(y-x))
= f(x)G(x) + \phi(y)(y-x)G(x) + f(y)\Gamma(y)(y-x)
= fG(x) + H(y)(y-x),
\]
where \( H : U \rightarrow \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \) is the function given by
\[
H(x) = f(x)\Gamma(y) + G(x)\phi(x).
\]
Continuity of \( H \) follows by application of the sum and product rules for continuous functions. By applying Hadamard’s lemma again, we conclude that \( fG \) is differentiable at \( x \), with total derivative
\[
D(fG)(x) = H(x) = f(x)\Gamma(x) + G(x)\phi(x) = f(x)DG(x) + G(x)Df(x).
\]
2. Because $f$ and $G$ are differentiable by assumption, one can write
\[ f(x + h) = f(x) + Df(x)h + R_f(x + h) \]
and
\[ G(x + h) = G(x) + DG(x)h + R_G(x + h). \]
Here $R_f : U \to \mathbb{R}$ and $R_G : U \to \mathbb{R}^n$ satisfy
\[
\lim_{h \to 0} \frac{R_f(x + h)}{\|h\|} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{R_G(x + h)}{\|h\|} = 0.
\]
By working out the product of these two expressions, one obtains
\[ (fG)(x + h) = f(x)G(x) + (f(x)DG(x)h + Df(x)hG(x)) + R_{fG}(x + h), \]
where the final term reads
\[ R_{fG}(x + h) = Df(x)hDG(x)h + R_f(x + h)G(x) + f(x + h)R_G(x + h). \]
Since $h \mapsto G(x)$ and $h \mapsto f(x + h)$ are continuous functions, we obviously have
\[
\lim_{h \to 0} \frac{R_f(x + h)}{\|h\|} = 0 \quad \text{and} \quad \lim_{h \to 0} f(x + h) \frac{R_G(x + h)}{\|h\|} = 0.
\]
For the first term, we can make the estimate
\[
\frac{|Df(x)h| \|DG(x)h\|}{\|h\|} \leq \frac{|Df(x)| \|DG(x)\| \|h\|^2}{\|h\|} = \|Df(x)\| \|DG(x)\| \|h\|,
\]
so this also vanishes in the limit for $h \to 0$. We conclude that
\[
\lim_{h \to 0} \frac{R_{fG}(x + h)}{\|h\|} = 0.
\]
Hence, $fG$ is differentiable and its total derivative is given by
\[
D(fG)(x)h = f(x)DG(x)h + Df(x)hG(x)
= (f(x)DG(x) + G(x)Df(x))h.
\]

3. One can use the fact that an $\mathbb{R}^n$-valued function is differentiable if and only if all of its components are.

For $1 \leq i \leq n$, the $i$-th component of $fG$ is given by $(fG)_i(x) = f(x)G_i(x)$ and is a product of scalar functions. Both $f$ and $G_i$ are differentiable by assumption, so one may conclude from the product rule that their product is as well, with total derivative
\[
D(fG)_i(x) = G_i(x)Df(x) + f(x)DG_i(x).
\]
Since each of its components are differentiable, the original function $fG$ is as well and its derivative is given by
\[
D(fG)(x)h = \begin{pmatrix} D(fG)_1(x)h \\ \vdots \\ D(fG)_n(x)h \end{pmatrix} = \begin{pmatrix} G_1(x)Df(x)h + f(x)DG_1(x)h \\ \vdots \\ G_n(x)Df(x)h + f(x)DG_n(x)h \end{pmatrix}.
\]
More concisely, we read off that $D(fG)(x) = G(x)Df(x) + f(x)DG(x)$. 

2
(c) (5 pt) Using (2) with \( f(x) = \|x\|^2 \), compute the total derivative \( DG(x) \) of the mapping (1) for \( x \in U \), where \( U = \mathbb{R}^n \setminus \{0\} \).

In our specific case, we have that \( f(x)G(x) = x \) for all \( x \in \mathbb{R}^n \setminus \{0\} \), so \( fG = \text{id} \). From this, it follows that

\[
DG(\text{id}) = GDf + fDG = D\text{id} = \text{id}.
\]

We know the derivative of \( f : x \mapsto \|x\|^2 \) to be \( Df(x)h = 2 \langle x, h \rangle = 2x^\top h \), so the above identity tells us that

\[
DG(x) = f(x)^{-1}(\text{id} - G(x) \cdot Df(x))
\]

\[
= \frac{1}{\|x\|^2} \left( \text{id} - \frac{x}{\|x\|^2} \cdot 2x^\top \right) = \frac{1}{\|x\|^2} A(x),
\]

where for \( x \in \mathbb{R} \setminus \{0\} \), \( A(x) \) denotes the matrix

\[
A(x) = I - 2 \frac{xx^\top}{\|x\|^2}.
\]

(d) (10 pt) Show that for \( x \in U \) holds \( DG(x) = \|x\|^{-2} A(x) \), where \( A(x) \) is represented by an orthogonal matrix, i.e. \( A^\top(x)A(x) = I \).

We recognise \( A(x) \) as the matrix representing a reflection in the plane perpendicular to \( x \). We will verify that this is an orthogonal transformation.

Because \( A^\top(x) = A(x) \), we see that

\[
A^\top(x)A(x) = \left( I - 2 \frac{xx^\top}{\|x\|^2} \right)^2 = I^2 - 4 \frac{x x^\top}{\|x\|^2} + 4 \frac{x x^\top}{\|x\|^2} \frac{x x^\top}{\|x\|^2} = I^2 - 4 \frac{x x^\top}{\|x\|^2} + 4 \frac{x x^\top}{\|x\|^2} \frac{x x^\top}{\|x\|^2}.
\]

Because \( x^\top x = \|x\|^2 \), the last two terms cancel out and we may conclude that \( A^\top(x)A(x) = I^2 = I \).

Exercise 2 (30 pt). Let \( a, b, c > 0 \) and let \( M \) be the ellipsoid in \( \mathbb{R}^3 \) defined as

\[
M = \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}.
\]

(a) (10 pt) Find the tangent space of \( M \) at \( x \in M \).

Introduce \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \) by

\[
g(x) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2},
\]

so that \( M = \left\{ x \in \mathbb{R}^3 : g(x) = 1 \right\} \). A simple computation shows that the derivative of \( g \) at \( x \in \mathbb{R}^3 \) reads

\[
Dg(x) = \begin{pmatrix} \frac{2x_1}{a^2} & \frac{2x_2}{b^2} & \frac{2x_3}{c^2} \end{pmatrix},
\]

which is non-zero for all \( x \neq 0 \). Hence \( g \) is a submersion at every point \( x \in M \) and its geometric tangent space at \( x \) is given by

\[
T_xM = \left\{ y \in \mathbb{R}^3 \mid Dg(x)(y - x) = 0 \right\} = \left\{ y \in \mathbb{R}^3 \mid Dg(x)y = 2 \right\}.
\]

For this have used that \( Dg(x)x = \frac{2x_1}{a^2}x_1 + \frac{2x_2}{b^2}x_2 + \frac{2x_3}{c^2}x_3 = 2g(x) = 2 \).
(b) \( (20 \text{ pt}) \) Compute the distance from the origin to the geometric tangent plane to \( M \) at an arbitrary point \( x \in M \).

The distance from the origin to the tangent plane at \( x \in M \) can be found through either a geometric argument or by applying the method of Lagrange multipliers.

1. The distance from the origin to the plane will be equal to the length of the component of \( x \in \tilde{T}_x M \) orthogonal to it. Since we know that \( \text{grad} \, g(x) = [Dg(x)]^T \) is orthogonal to the tangent space \( \tilde{T}_x M \), this length will be given by
   \[
d(0, \tilde{T}_x M) = \frac{\langle x, \text{grad} \, g(x) \rangle}{\|\text{grad} \, g(x)\|} = \frac{Dg(x)x}{\|Dg(x)\|}.
   \]
   We have already computed the numerator \( Dg(x)x = 2 \), and the denominator can be read off from equation (3). We thus obtain
   \[
d(0, \tilde{T}_x M) = \frac{x_1^2 + x_2^2 + x_3^2}{\sigma^2} - \frac{1}{2}.
   \]

2. One may also arrive at this answer through the method of Lagrange multipliers. The distance \( d(0, \tilde{T}_x M) \) is then obtained by minimising the function \( f : x \mapsto \|x\|^2 \) on the geometric tangent plane \( \tilde{T}_x M \). Since the plane \( \tilde{T}_x M \subseteq \mathbb{R}^3 \) is a closed subset of \( \mathbb{R}^3 \), \( f \) assumes a minimum on it at some point \( y_0 \in \tilde{T}_x M \) and the distance from the origin to the plane will be the square root of this minimum. (NB: The intersection \( \tilde{T}_x M \cap \overline{B(0, R)} \) is compact and non-empty for an appropriately chosen \( R > 0 \). The norm assumes a minimum on it, which is in fact a global minimum.)

The point \( y_0 \in \tilde{T}_x M \) will necessarily be a critical point for \( f \), which means that \( \text{grad} \, f(y_0) = 2y_0 \) is orthogonal to \( \tilde{T}_x M \), hence parallel to \( \text{grad} \, g(x) \). Let \( \lambda \in \mathbb{R} \) be such that \( y_0 = \lambda \text{grad} \, g(x) \), then we see that (since \( y_0 \in \tilde{T}_x M \))
   \[
   Dg(x)y_0 = \langle \text{grad} \, g(x), \lambda \text{grad} \, g(x) \rangle = \lambda \|\text{grad} \, g(x)\|^2 = 2.
   \]
   We derive that \( \lambda = 2\|\text{grad} \, g(x)\|^{-2} \) and that therefore
   \[
   \|y_0\| = \|\lambda \|\text{grad} \, g(x)\| = \frac{2}{\|\text{grad} \, g(x)\|} = \left(\frac{x_1^2 + x_2^2 + x_3^2}{\sigma^2}\right)^{-\frac{1}{2}}.
   \]
   This confirms our earlier conclusion.

3. The critical point described in part 2 also corresponds to a critical point for the Lagrange function
   \[
   L : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \quad (y, \lambda) \mapsto f(y) - \lambda h(y),
   \]
   where \( f(y) = \|y\|^2 \) and \( h(y) = Dg(x)y - 2 \).
   Since \( Df(y) = 2y^r \) and \( Dh(y) = Dg(x) \), the equation \( DL(y, \lambda) = 0 \) becomes
   \[
   DL(y) = (Df(y) - \lambda Dh(y), h(y)) = (2y^r - \lambda Dg(x), Dg(x)y - 2) = 0.
   \]
   Solving this system of equations essentially comes down to following the steps from option 2.
Exercise 3. (40 pt) Here, we will study a representation of the Möbius Strip in $\mathbb{R}^3$.

(a) (5 pt) Let $D = \{(\theta, t) \in \mathbb{R}^2 : -\pi < \theta < \pi, -1 < t < 1\}$ and let $\Phi : D \to \mathbb{R}^3$ be defined by

$$\Phi(\theta, t) = \begin{pmatrix} 2 + t \cos \left(\frac{\theta}{2}\right) \cos \theta \\ 2 + t \cos \left(\frac{\theta}{2}\right) \sin \theta \\ t \sin \left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Prove that $\Phi$ is an immersion at any point in $D$.

The function $\Phi$ is clearly $C^\infty$, and we can explicitly compute its derivative

$$D\Phi(\theta, t) = (\partial_\theta \Phi(\theta, t) \quad \partial_t \Phi(\theta, t))$$

$$= \begin{pmatrix} -\frac{1}{2} t \sin \left(\frac{\theta}{2}\right) \cos \theta - (2 + t \cos \left(\frac{\theta}{2}\right)) \sin \theta & \cos \left(\frac{\theta}{2}\right) \cos \theta \\ -\frac{1}{2} t \sin \left(\frac{\theta}{2}\right) \sin \theta + (2 + t \cos \left(\frac{\theta}{2}\right)) \cos \theta & \cos \left(\frac{\theta}{2}\right) \sin \theta \\ t \cos \left(\frac{\theta}{2}\right) & \sin \left(\frac{\theta}{2}\right) \end{pmatrix}.$$

There are at least three ways to verify that $D\Phi(\theta, t)$ is injective for all $(\theta, t) \in D$, so that $\Phi$ is an immersion.

1. One can compute the determinant of the upper $2 \times 2$ block of $D\Phi(\theta, t)$. This determinant equals

$$-(2 + t \cos \left(\frac{\theta}{2}\right)) \cos \left(\frac{\theta}{2}\right).$$

This is non-zero for all $(\theta, t) \in D$, meaning that $D\Phi(\theta, t)$ has rank 2 and that $\Phi$ is an immersion.

2. One can also decompose

$$D\Phi(\theta, t) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} t \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right) \\ 2 + t \cos \left(\frac{\theta}{2}\right) \cos \theta \\ \frac{1}{2} t \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta}{2}\right) \end{pmatrix}.$$ 

Since $2 + t \cos \left(\frac{\theta}{2}\right) > 0$ for $(\theta, t) \in D$, the two columns of the $3 \times 2$-matrix on the second line are linearly independent. Because the square matrix that was factored out is invertible, we conclude that $D\Phi(\theta, t)$ is injective and that $\Phi$ is therefore an immersion.

3. Another option is calculating the cross product $\partial_\theta \Phi(\theta, t) \times \partial_t \Phi(\theta, t)$. The third component of this cross product is

$$-(2 + t \cos \left(\frac{\theta}{2}\right)) \cos \left(\frac{\theta}{2}\right) \sin \theta + \cos \left(\frac{\theta}{2}\right) \cos \theta = -(2 + t \cos \left(\frac{\theta}{2}\right)) \cos \left(\frac{\theta}{2}\right).$$

This is non-zero for all $(\theta, t) \in D$, which means that the columns of $D\Phi(\theta, t)$ are linearly independent. We conclude that $D\Phi(\theta, t)$ has rank 2 and that $\Phi$ is an immersion.

(b) (10 pt) Show that $\Phi : D \to \Phi(D)$ is invertible and that the inverse mapping is continuous. Use this to conclude that $V = \Phi(D)$ is a $C^\infty$ submanifold in $\mathbb{R}^3$ of dimension 2.

For $(x, y) \in \mathbb{R}^2$ of the form $(x, y) = \rho (\cos \phi, \sin \phi)$ with $\rho > 0$ and $\phi \in ]-\pi, \pi[$, one can recover $\rho = \sqrt{x^2 + y^2}$ and $\phi = 2 \arctan(\frac{y}{x + \sqrt{y^2}})$. We therefore define

$$\rho : \mathbb{R}^2 \setminus \{(0, 0)\} \to ]0, \infty[, \quad \text{and} \quad \phi : \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\} \to ]-\pi, \pi[.$$
by setting
\[ \rho(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad \phi(x, y) = 2 \arctan \left( \frac{y}{\rho(x, y) + x} \right). \]

Since all functions involved are smooth on their domain, \( \rho \) and \( \phi \) are \( C^\infty \) as well.

If \((x, y, z) = \Phi(\theta, t)\), then we see that \( \theta = \phi(x, y) \) and \( 2 + t \cos(\frac{1}{2} \theta) = \rho(x, y) \), from which \( t \) can also be obtained since \( \cos(\frac{1}{2} \theta) \neq 0 \). This leads us to conclude that the map \( \Psi : \mathbb{R}^3 \setminus \{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\} \to \pi, \pi \times \mathbb{R} \) such that

\[ \Psi(x, y, z) = \left( \frac{\phi(x, y)}{\rho(x, y) + 2 \cos(\frac{1}{2} \phi(x, y))} \right) \]

is a left-inverse of \( \Phi \), i.e. \( \Psi \circ \Phi = \text{id} : D \to D \). We deduce that \( \Phi \) is injective and that its inverse is the restriction \( \Psi|_{\Phi(D)} : \Phi(D) \to D \).

Since we have described it as a composition of continuous functions, \( \Psi \) is also continuous, as is the restriction \( \Psi|_{\Phi(D)} : \Phi(D) \to D \). We conclude that \( \Phi \) is a \( C^\infty \) embedding and that its image \( \Phi(D) \) is therefore a 2-dimensional \( C^\infty \) submanifold of \( \mathbb{R}^3 \).

(c) (5 pt) Prove that any point \( x \in V \) satisfies \( g(x) = 0 \), where \( g : \mathbb{R}^3 \to \mathbb{R} \) is defined by

\[ g(x) = 4x_2 + 4x_1x_3 - x_2(x_1^2 + x_2^2 + x_3^2) + 2x_3(x_1^2 + x_2^2). \] (4)

Notice that each term in \( g \) has factor \( (2 + t \cos(\frac{1}{2} \theta)) \). This implies

\[ g = (2 + t \cos(\frac{1}{2} \theta)) \left[ 4 \sin \theta + 4t \cos \theta \sin(\frac{1}{2} \theta) - \sin \theta \left( 4 + 4t \cos(\frac{1}{2} \theta) + t^2 \right) + 2t \sin(\frac{1}{2} \theta) \right] \]
\[ = (2 + t \cos(\frac{1}{2} \theta)) \left[ 4t \cos \theta \sin(\frac{1}{2} \theta) - \sin \theta \cos(\frac{1}{2} \theta) \right. \]
\[ - t^2 \sin \theta + 4t \sin(\frac{1}{2} \theta) + 2t^2 \sin(\frac{1}{2} \theta) \cos(\frac{1}{2} \theta) \right] \]
\[ = (2 + t \cos(\frac{1}{2} \theta)) \left[ -4t \sin(\frac{1}{2} \theta) - t^2 \sin \theta + 4t \sin(\frac{1}{2} \theta) + t^2 \sin \theta \right] = 0, \]

since
\[ 2 \sin(\frac{1}{2} \theta) \cos(\frac{1}{2} \theta) = \sin \theta \]

and
\[ \cos \theta \sin(\frac{1}{2} \theta) - \sin \theta \cos(\frac{1}{2} \theta) = \sin(\frac{1}{2} \theta - \theta) = - \sin(\frac{1}{2} \theta). \]

We conclude that \( g(\Phi(\theta, t)) = 0 \) for all \( (\theta, t) \in D \).

(d) (10 pt) The Möbius strip is the closure \( M = \overline{V} \) of \( V \) in \( \mathbb{R}^3 \). Verify that the circle \( S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 4 \text{ and } x_3 = 0\} \) belongs to \( M \). Give a parametrization of \( S \) by \( \theta \in ]-\pi, \pi[ \). Prove that \( g \) introduced by (4) is a submersion at any point \( x \in S \) except for \( x = (-2, 0, 0) \).

One can parametrise the circle \( S \) by \( f : ]-\pi, \pi[ \to \mathbb{R}^3, \theta \mapsto (2 \cos \theta, 2 \sin \theta, 0) \). Note that \( f(] - \pi, \pi[) \subseteq \Phi(D) \) because \( f(\theta) = \Phi(\theta, 0) \) for \( \theta \in ] - \pi, \pi[ \).

The fact that \( f \) is continuous then tells us that

\[ f(] - \pi, \pi[) = f(] - \pi, \pi[) \subseteq f(] - \pi, \pi[) \subseteq \overline{V} = M, \]

where \( ]-\pi, \pi[ = ]-\pi, \pi[ \) denotes the closure of \( ]-\pi, \pi[ \) in \( ]-\pi, \pi[ \).
One way to derive this is by writing $\pi = \lim_{n \to -\infty} a_n$ for some sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in ]-\pi, \pi[$, so that $f(\pi) = \lim_{n \to -\infty} f(a_n)$ by the continuity of $f$. From this we conclude that $f(\pi)$ is a limit point of $f([-\pi, \pi]) \subseteq V$ and is therefore in the closure $M = \overline{V}$.

The gradient of $g$ can easily be computed, and reads

$$
\nabla g(x) = \begin{pmatrix}
4x_3 - 2x_1 x_2 + 4x_1 x_3 \\
4 - (x_1^2 + x_2^2 + x_3^2) - 2x_2^2 + 4x_3 x_2 \\
x_1 - 2x_2 x_3 + 2(x_1^2 + x_2^2)
\end{pmatrix}
$$

By plugging in $x = f(\theta)$, we obtain the expression

$$
\nabla g(f(\theta)) = \begin{pmatrix}
-8 \cos \theta \sin \theta \\
4 - 4 - 8 \sin^2 \theta \\
8 \cos \theta + 8
\end{pmatrix} = 4 \begin{pmatrix}
-\sin(2 \theta) \\
\cos(2 \theta) - 1 \\
2 (\cos \theta + 1)
\end{pmatrix}.
$$

The last component is non-zero for all $\theta \in ]\pi, \pi[$, while for $\theta = \pi$ all components vanish. Thus, $g$ is a submersion at every point of $S$ except for $f(\pi) = (-2, 0, 0)$.

This shows that $V$ is a submanifold at every point in $S \cap V$, corroborating the conclusion from part (b).

(e) \hspace{1cm} (10 pt) Show that $n_0 = (0, 0, 1) \in \mathbb{R}^3$ is orthogonal to the tangent space $T_{\Phi(0,0)}V$. Compute a continuous vector-valued function $n : ]-\pi, \pi[ \to \mathbb{R}^3$ such that $n(0) = n_0$ and for all $-\pi < \theta < \pi$ the vector $n(\theta) \in \mathbb{R}^3$ is orthogonal to $T_{\Phi(\theta,0)}V$ while $\|n(\theta)\| = 1$. Verify that $\lim_{\theta \to -\pi} n(\theta) = -\lim_{\theta \to -\pi} n(\theta)$.

Here again several approaches are possible.

1. Since we have shown that the function $g$ is a submersion at $x = \Phi(\theta, 0) = f(\theta)$ for $\theta \in ]-\pi, \pi[$ and $V \subseteq g^{-1}(\{0\})$, we also know that the gradient $\nabla g(x)$ is normal to the tangent space $T_{\Phi(\theta,0)}V$. Because $\nabla g(f(0)) = (0, 0, 16)$, it follows that also $n_0 = (0, 0, 1)$ is orthogonal to $T_{\Phi(0,0)}V$.

The function $n$ described in the exercise is obtained by normalising the vectors $\nabla g(f(\theta))$ for $\theta \in ]-\pi, \pi[$ and setting

$$
n(\theta) = \frac{\nabla g(f(\theta))}{\|\nabla g(f(\theta))\|} = \frac{1}{4|\cos(\frac{\theta}{2})|} \begin{pmatrix}
-\sin(2 \theta) \\
\cos(2 \theta) - 1 \\
2 (\cos \theta + 1)
\end{pmatrix}.
$$

A few trigonometric identities have been applied to obtain the final, simplified expression:

\[
\begin{align*}
\sin^2(2 \theta) + (\cos(2 \theta) - 1)^2 + 4 (\cos \theta + 1)^2 \\
= \sin^2(2 \theta) + \cos^2(2 \theta) - 2 \cos(2 \theta) + 1 + 4 \cos^2 \theta + 8 \cos \theta + 4 \\
= 6 - 2(\cos^2 \theta - \sin^2 \theta) + 4 \cos^2 \theta + 8 \cos \theta \\
= 8 + 8 \cos \theta = 16 \cos^2(\frac{\theta}{2}).
\end{align*}
\]

We note that $|\cos(\frac{\theta}{2})| = \cos(\frac{\theta}{2})$ for $-\pi \leq \theta \leq \pi$, so that the limits $\lim_{\theta \to \pm \pi} n(\theta)$ can be
obtained by applying l’Hôpital’s rule:

\[
\lim_{\theta \to \pm \pi} n(\theta) = \lim_{\theta \to \pm \pi} \frac{1}{4 \cos(\frac{1}{2} \theta)} \left( \frac{-\sin(2 \theta)}{\cos(2 \theta) - 1} \right) \left( \frac{\cos(2 \theta) - 1}{2 (\cos \theta + 1)} \right)
\]

\[
= \lim_{\theta \to \pm \pi} \frac{1}{4 \cos(\frac{1}{2} \theta)} \frac{d}{d\theta} \left( \frac{-\sin(2 \theta)}{\cos(2 \theta) - 1} \right) \left( \frac{\cos(2 \theta) - 1}{2 (\cos \theta + 1)} \right)
\]

\[
= \lim_{\theta \to \pm \pi} \frac{1}{-2 \sin(\frac{1}{2} \theta)} \left( \begin{array}{c}
-2 \cos(2 \theta) \\
-2 \sin(2 \theta) \\
-2 \sin \theta
\end{array} \right)
\]

This is just the limit of a continuous function, so we read off that

\[
\lim_{\theta \to \pm \pi} n(\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \lim_{\theta \to -\pi} n(\theta) = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

2. A somewhat different approach involves the cross product \( \partial_\theta \Phi(\theta, t) \times \partial_t \Phi(\theta, t) \) of the partial derivatives of part (a). Because \( \Phi \) is an immersion, this cross-product is non-vanishing for every \( (\theta, t) \in D \), and is orthogonal to the tangent space \( T_{\Phi(\theta,t)} \).

Since at \( \partial_\theta \Phi(0,0) = (0,2,0) \) and \( \partial_t \Phi(0,0) = (1,0,0) \), we have \( \partial_\theta \Phi(0,0) \times \partial_t \Phi(0,0) = (0,0,-2) \) and we can again conclude that \( n_0 = (0,0,1) \) is orthogonal to \( T_{\Phi(0,0)} V \).

Because \( \partial_\theta \Phi(0,0) \times \partial_t \Phi(0,0) \) and \( n_0 \) are pointing in opposite directions, an additional minus sign needs to be introduced in the definition of \( n \), so that

\[
n(\theta) = -\frac{\partial_\theta \Phi(\theta,0) \times \partial_t \Phi(\theta,0)}{\| \partial_\theta \Phi(\theta,0) \times \partial_t \Phi(\theta,0) \|}.
\]

This will lead to the same answer.

(f) (Bonus: 5 pt) Sketch the set \( M \) and describe its geometry.

The Möbius strip \( M \) is a smooth 2-dimensional connected manifold with boundary in \( \mathbb{R}^3 \). It is similar to a cylinder in the sense that it can be described as the union of a continuous family of line segments over the circle, but these line segments are gradually twisted as one goes around the circle. This happens in such a way that if one follows a line segment around the circle once, its end points are interchanged. (It is a non-trivial fibre bundle.)
The Möbius strip is non-orientable, which can be expressed by saying that it has only ‘one side’. This was demonstrated in part (e), where a vector normal to the surface was continuously transported around the loop once and ended up on the ‘other side’.