Measure and Integration: Final 2015-16

(1) Consider the measure space \([0, 1], \mathcal{B}([0, 1]), \lambda\) where \(\lambda\) is Lebesgue measure on \([0, 1]\). Define \(u_n(x) = \frac{n^2 x^2}{1 + n^2 x^2}\) for \(x \in [0, 1]\) and \(n \geq 1\). Show that
\[
\lim_{n \to \infty} \int_{[0, 1]} \frac{n^2 x^2}{1 + n^2 x^2} \, d\lambda(x) = 1/2.
\]
(1 pt)

(2) Suppose \(\mu\) and \(\nu\) are finite measures on \((X, \mathcal{A})\). Show that there exists a function \(f \in \mathcal{L}_1^1(\mu)\), and a set \(A_0 \in \mathcal{A}\) with \(\mu(A_0) = 0\) such that
\[
\nu(E) = \int_E f \, d\mu + \nu(A_0 \cap E),
\]
for all \(E \in \mathcal{A}\). (1.5 pts)

(3) Consider the measure space \([0, 1), \mathcal{B}([0, 1)), \lambda\) where \(\lambda\) is Lebesgue measure on \([0, 1)\). Let \(D_1 = [0, 1/2)\) and \(D_k = \left[\sum_{i=1}^{k-1} 2^{-i}, \sum_{i=1}^{k} 2^{-i}\right), k \geq 2\). Define \(u(x) = \sqrt{2k-1}\) for \(x \in D_k\), \(k \geq 1\). Determine the values of \(p \in [1, \infty)\) such that \(u \in \mathcal{L}_p^p(\lambda)\). In case \(u \in \mathcal{L}_p^p(\lambda)\), find the value of \(\|u\|_p\). (2 pts.)

(4) Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, and Let \((u_j) \subseteq \mathcal{L}_1^1(\mu)\). Suppose \((u_j)\) converges to \(u\) \(\mu\) a.e., and that the sequence \((u_j^-)\) is uniformly integrable. Prove that
\[
\liminf_{n \to \infty} \int u_n \, d\mu \geq \int u \, d\mu.
\]
(2 pts)

(5) Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, and assume \(u \in \mathcal{M}^+(\mathcal{A})\). Let \(\phi : [0, \infty) \to \mathbb{R}\) be continuously differentiable (i.e. \(\phi'\) exists and is continuous) such that \(\phi(0) = 0\) and \(\phi' \geq 0\) for all \(t \geq 0\). Show that
\[
\int_X \phi \circ u(x) \, d\mu = \int_{[0, \infty)} \phi'(t) \mu\{\{x \in X : u(x) \geq t\}\} \, d\lambda(t).
\]
Conclude that if \(u \in \mathcal{L}_p^p(\mu)\), then
\[
\int_X u^p \, d\mu = p \int_{[0, \infty)} t^{p-1} \mu\{\{x \in X : u(x) \geq t\}\} \, d\lambda(t).
\]
(2 pts)

(6) Let \((X, \mathcal{A}, \mu)\) be a measure space and \(f \in \mathcal{L}_1^1(\mu) \cap \mathcal{L}_2^2(\mu)\).
(a) Show that \(f \in \mathcal{L}_p^p(\mu)\) for all \(1 \leq p \leq 2\). (1 pt)
(b) Prove that \(\lim_{p \uparrow 1} \|f\|_p^p = \|f\|_1\). (1 pt)