Measure and Integration: Retake Final 2015-16

(1) Let $(X, A, \mu)$ be a finite measure space, and $f \in \mathcal{M}(A)$. Show that for every $\epsilon > 0$, there exists a set $A \in \mathcal{A}$ and $k \geq 1$ such that $\mu(A) < \epsilon$ and $|f(x)| \leq k$ for all $x \in A^c$. (1 pt)

(2) Consider the measure space $[0, 1], B([0, 1]), \lambda$ where $\lambda$ is Lebesgue measure on $[0, 1]$. Define $u_n(x) = \frac{nx}{1 + n^2 x^2}$ for $x \in [0, 1]$ and $n \geq 1$. Show that

$$\lim_{n \to \infty} \int_{[0, 1]} \frac{nx}{1 + n^2 x^2} d\lambda(x) = 0.$$ (1.5 pts)

(3) Let $\mu$ and $\nu$ be finite measures on $(X, \mathcal{A})$. Show that $\mu$ and $\nu$ are mutually singular if and only if for every $\epsilon > 0$, there exists a set $E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and $\nu(E^c) < \epsilon$. (2 pts)

(4) Let $(X, A, \mu)$ be a measure space, and $(u_n)_n \subseteq L^p(\mu)$ converging in $L^p(\mu)$ to a function $u \in L^p(\mu)$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$, then $\int_A |u_n|^p d\mu < \epsilon$ for all $n \geq 1$. (2 pts)

(5) Consider the measure space $([0, \infty), B([0, \infty)), \lambda)$, where $B([0, \infty))$ is the Borel $\sigma$-algebra, and $\lambda$ is Lebesgue measure on $[0, \infty)$. Let $f(x, y) = ye^{-(1+x^2)y^2}$ for $0 \leq x, y < \infty$.

(a) Show that $f \in L^1(\lambda \times \lambda)$, and determine the value of $\int_{[0, \infty) \times [0, \infty)} f d(\lambda \times \lambda)$. (1 pt)

(b) Prove that $\int_{[0, \infty) \times [0, \infty)} f d(\lambda \times \lambda) = \left( \int_{[0, \infty)} e^{-x^2} d\lambda(x) \right)^2$. Use part (a) to deduce the value of $\int_{[0, \infty)} e^{-x^2} d\lambda(x)$. (1 pt)

(6) Let $(X, A, \mu)$ be a $\sigma$-finite measure space, and Let $(u_j)_j \subseteq L^p(\mu)$, $p \geq 1$. Suppose $(u_j)_j$ converges to $u \mu$ a.e., and that the sequence $\left( (u_j^p)^- \right)$ is uniformly integrable. Prove that

$$\liminf_{n \to \infty} \int u_n^p d\mu \geq \int u^p d\mu.$$ (1.5 pts)