Solution to 1

(a) Let $B_1, B_2 \in \mathcal{B}$. If one of $B_1, B_2$ equals $\mathbb{R}$, then obviously $B_1 \cap B_2 \in \mathcal{B}$. Assume that $B_1, B_2$ are not equal to $\mathbb{R}$. Then $B_j = [n_j, a_j)$, with $n_1, n_2 \in \mathbb{Z}$ and $a_1, a_2 \in \mathbb{R}$. It is now readily seen that $B_1 \cap B_2 = [n, b)$ with $n = \max(m_1, m_2)$ and $b = \min(a_1, a_2)$. Hence $B_1 \cap B_2 \in \mathcal{B}$. This shows that $\mathcal{B}$ is a topology basis. Since $\bigcup_{m<0}[m,0) = (\infty, 0) \notin \mathcal{B}$, we see that $\mathcal{B}$ is not closed under unions. It follows that $\mathcal{B}$ is not a topology.

(b) Let $U \in \mathcal{T}$ contain $\frac{1}{2}$. Then there exists $[m,a) \in \mathcal{B}$ with $\frac{1}{2} \in [m,a) \subset U$. We must have $m \leq 0$ and $a > \frac{1}{2}$, hence $[0,a) \subset U$. In particular, $0 \in U$. It follows that 0 and $\frac{1}{2}$ cannot be separated by open neighborhoods. Hence, $\mathcal{T}$ is not Hausdorff.

(c) The subset $\mathcal{B}_0 \subset \mathcal{B}$ consisting of all intervals $[m,q)$ with $m \in \mathbb{Z}$ and $q \in \mathbb{Q}$ is countable. Moreover, if $a > 0$ then $[m,a) = \cup_{q \in \mathbb{Q}, q < a}[m,q)$, so $\mathcal{B}_0$ is a countable basis for $\mathcal{T}$. It follows that $\mathcal{T}$ is second countable.

(d) A non-empty basis element $[m,a) \in \mathcal{B}$ is contained in $A$ if and only if $m \geq 0$ and $a \leq \frac{1}{2}$. The latter is equivalent to $m = 0$ and $a \leq \frac{1}{2}$. The union of these sets is $\text{Int}(A) = [0, \frac{1}{2})$.

The condition $x \notin \overline{A}$ is equivalent to the existence of $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ with $x \in [m,a)$ and $[m,a) \cap A = \emptyset$. The latter condition forces $m \geq 1$ or $a < -\frac{1}{2}$ and we see that $x \notin \overline{A}$ implies $x \in [1,\infty)$ or $x \in (-\infty,-\frac{1}{2})$. Conversely, if $x \in [1,\infty)$ or $x \in (-\infty,-\frac{1}{2})$ then either $x \in [1,a)$ for $a > 1$ or $x \in [m,-\frac{1}{2})$ for $m \leq -1$. In both cases, there exist $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ such that $x \in [m,a)$ and $[m,a) \cap A = \emptyset$. We conclude that $\overline{A}$ equals the complement of $[1,\infty) \cup (-\infty, -\frac{1}{2})$ which equals $[-\frac{1}{2},1)$.

(e) Assume that $0 < r < 1$. Any open subset $U$ of $[0,r]$ containing $r$ must contain a subset of the form $[0,r] \cap [m,a)$, for $m \leq r < a$. The latter implies $m \leq 0$ and $a > r$ hence $[0,r] \subset [0,r] \cap [0,a) \subset U$ hence $U = [0,r]$. This implies that $[0,r]$ cannot be written as the union of two disjoint non-empty open subsets. Hence $[0,r]$ is connected.

Now assume that $r \geq 1$. Then $[1,r] = [0,r] \cap [1,r+1)$ hence $[1,r]$ is open and non-empty in $[0,r]$. Obviously, $[0,1)$ is open and non-empty in $[0,r]$ and $[0,r]$ is the disjoint union of $[0,1)$ and $[1,r]$. It follows that $[0,r]$ is not connected.
Solution to 2

(a) We assume that both $X$ and $Y$ are Hausdorff. Let $a, b \in X \times Y$ be two points such that $a \neq b$. Write $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then we may as well assume that $a_1 \neq b_1$. By the Hausdorff property of $X$ there exist open subsets $U, V \subset X$ such that $a_1 \in U, a_2 \in V$ and $U \cap V = \emptyset$. Now $U \times Y$ and $V \times Y$ are open subsets of $X \times Y$ containing $a$ and $b$ respectively, and

$$U \times Y \cap V \times Y = (U \cap V) \times Y = \emptyset.$$ 

It follows that the product is Hausdorff.

(b) For the converse, assume that $X \times Y$ is Hausdorff. Let $a_1, b_1 \in X$ be distinct points. Select a point $y \in Y$ then $(a_1, y)$ and $(b_1, y)$ are distinct points in $X \times Y$. By the Hausdorff property, there exist open subsets $W_1, W_2$ in $X \times Y$ such that $(a_1, y) \in W_1$, $(b_1, y) \in W_2$ and $W_1 \cap W_2 = \emptyset$. Since $W_1$ is open, there exists an open subset $U_1 \supset a_1$ of $X$ such that $U_1 \times \{ y \} \subset W_1$. Likewise, there exists an open subset $U_2 \supset b_1$ of $X$ such that $U_2 \times \{ y \} \subset W_2$. We now observe that

$$(U_1 \cap U_2) \times \{ y \} = U_1 \times \{ y \} \cap U_2 \times \{ y \} \subset W_1 \cap W_2 = \emptyset.$$ 

It follows that $U_1 \cap U_2 = \emptyset$. We conclude that $a_1, b_1$ are separated in $X$. Hence, $X$ is Hausdorff. In a similar way, it follows that $Y$ is Hausdorff.

Solution to 3

1. We first show that ‘(2) $\Rightarrow$ (1)’. Let $g : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f \leq g$. Let $a \in X$. Let $U = g^{-1}((\infty, g(a) + 1))$. Then by continuity of $g$ it follows that $U$ is open. Clearly $a \in U$. Furthermore, $g \leq g(a) + 1$ on $U$. It follows that $f \leq M$ on $U$, with $M = g(a) + 1$.

2. We now address the converse implication ‘(1) $\Rightarrow$ (2)’. Assume that $f$ is locally bounded. Since $X$ is locally compact Hausdorff and second countable, it is paracompact.

For every $a \in X$ there exists an open neighborhood $V_a$ of $a$ such that $f$ is bounded on $V_a$ by a suitable constant $M_a > 0$. Let $\mathcal{V}$ be a collection of such open neighborhoods $V_a$, for $a \in X$.

**First reasoning.** Then by paracompactness, $\mathcal{V}$ has a locally finite refinement $\mathcal{U} = \{ U_i \mid i \in I \}$. For every $i \in I$ the neighborhood $U_i$ is contained in a neighborhood $V_{a(i)}$ for a suitable $a(i) \in X$, hence $f$ is bounded by $M_{a(i)} > 0$ on the neighborhood $U_i$.

Again by paracompactness, there exists a partition of unity $\{ \eta_i \mid i \in I \}$, with $\text{supp} \eta_i \subset U_i$ for all $i \in I$. The function $M_{a(i)} \eta_i$ is continuous and has support contained in $U_i$. 

2
Second reasoning. By paracompactness, there exists a partition of unity \( \{ \eta_i \mid i \in I \} \) which is subordinated to \( \mathcal{V} \). Thus, for every \( i \in I \) there exists a \( V_{a(i)} \in \mathcal{V} \) such that \( \text{supp} \eta_i \subset V_{a(i)} \). It follows that the function \( f \) is on \( V_{a(i)} \) bounded by a constant \( M_{a(i)} > 0 \). The function \( f \) is continuous and has support contained in \( \text{supp} \eta_i \).

From both reasonings given above, it follows that for all \( i \), we have \( f \eta_i \leq M_{a(i)} \eta_i \) on \( \text{supp} \eta_i \) hence on \( X \). Furthermore, the sum \( g := \sum_i M_{a(i)} \eta_i \) is a locally finite sum of continuous functions, hence continuous.

Finally, for \( x \in X \) we have

\[
f(x) = \sum_{i \in I} f(x) \eta_i(x) \leq \sum_{i \in I} M_{a(i)} \eta_i(x) = g(x).
\]

Solution to 4

(a) For \( \gamma_1, \gamma_2 \in \Gamma \) we have

\[
\rho_{\gamma_1 \gamma_2} = (\alpha_{\gamma_1 \gamma_2}, \beta_{\gamma_1 \gamma_2}) = (\alpha_1 \alpha_2, \beta_1 \beta_2) = \rho_1 \rho_2,
\]

and \( \rho_1 = (\alpha_1, \beta_1) = (\text{id}_{S^1}, \text{id}_{S^1}) = \text{id}_{S^1 \times S^1} \). Therefore, \( \rho \) defines an action of \( \Gamma \) on \( S^1 \times S^1 \). For a given \( \gamma \) the maps \( \alpha_\gamma, \beta_\gamma : S^1 \to S^1 \) are continuous, hence so is \( \rho_\gamma = (\alpha_\gamma, \beta_\gamma) : S^1 \times S^1 \to S^1 \times S^1 \). It follows that \( \rho \) is an action by homeomorphisms on \( S^1 \times S^1 \).

(b) Let \( p = (x, y) \). Then the orbit \( \Gamma p \) consists of \( 1p = p = (x, y) \) and \( gp = (−x, −y_1, y_2) \). Since \( x \neq −x \), each orbit consists of precisely two points.

(c) It is obvious that \( f \) is continuous. We claim that \( f \) is injective. Indeed, let \( f(s, y) = f(s', y') \), for \( (s, y), (s', y') \in [0, 1] \times S^1 \). Then \( y = y' \) and \( \cos s \pi = \cos s' \pi \) and \( \sin s \pi = \sin s' \pi \). Since \( \pi s \in [0, \pi] \), the latter two conditions imply that \( s = s' \). Hence, \( f \) is injective. Finally, since \( [0, 1] \times S^1 \) is compact and \( S^1 \times S^1 \) Hausdorff, it follows that \( f \) is a topological embedding.

(d) Let \( z \in S^1 \times S^1 / \Gamma \) and select \( (x, y) \in S^1 \times S^1 \) such that \( \pi(x, y) = z \). We note that \( x = (\cos \pi s, \sin \pi s) \) for a unique \( s \in [0, 2] \).

If \( s \in [0, 1] \) then \( \pi(x, y) = F(s, y) \) and we are done.

If \( s > 1 \), then \( −x = (\cos \pi (s − 1), \sin \pi (s − 1)) \) hence

\[
\pi(x, y) = \pi(g(x, y)) = \pi(−x, \beta_\gamma y) = \pi f(s − 1, \beta_\gamma y) = F(s − 1, \beta_\gamma y).
\]

Since \( (s − 1, y) \in [0, 1] \times S^1 \), we see that \( F \) is surjective.

(e) We observe that the map \( F \) induces an injective map \( \tilde{F} : [0, 1] \times S^1 / \sim \to S^1 \times S^1 / \Gamma \) such that \( \tilde{F} \circ \text{pr} = F \). Here \( \text{pr} : [0, 1] \times S^1 \to [0, 1] \times S^1 / \sim \) is the canonical
We will now describe the fibers of \( F \) since \( S \) is homeomorphic to the Klein bottle. From this it is clear that \( s = 0 \) and \( \sin \pi s = 0 \). Hence \( s = 0 = s' \) and then \( \cos \pi s' = 1 = \cos \pi s \), contradiction.

We thus see that \( \gamma = 1 \) hence \( \cos \pi s', \sin \pi s', y' = (\cos \pi s, \sin \pi s, y) \). Hence, \( s = s' \) and \( y = y' \) and the injectivity follows.

We will now describe the fibers of \( F \). From (*) we obtain that

\[
F(1,y) = F(0, \beta_s y).
\]

so that the fiber of \( F(1,y) \) contains \((0, \beta_s y)\) and \((1,y)\). Since \( F \) is injective on \([0,1) \times S^1\), we see that the fiber of \( F(1,y) \) cannot contain any other point. Again by injectivity of \( F \) on \([0,1)\) it follows that the fiber of \( F(s,y) \) for \( s \notin \{0,1\} \) can only contain the point \((s,y)\).

We see that for two distinct points \((s,y), (s',y')\) with \( s \leq s' \) we have \((s,y) \sim (s',y')\) if and only if \( s = 0, s' = 1 \) and \( y' = \beta_s y = (-y_1, y_2) \).

From this it is clear that \([0,1] \times S^1/\sim \) equipped with the quotient topology is homeomorphic to the Klein bottle.

In particular, it follows that \( S^1 \times S^1/\Gamma \) is homeomorphic to the Klein bottle.