

Solution 1

- (a) If $U \in \mathcal{T}$ and $c \in \mathbb{R}$ the pre-image $T_c^{-1}(U)$ belongs to \mathcal{T} by continuity of T_c . The pre-image is the set of $x \in \mathbb{R}$ such that $x + c \in U$, which equals $U + (-c)$. Thus, if we take $c = -a$ we see that $U + a \in \mathcal{T}$.
- (b) If $U \in \mathcal{T}$ then also the set $\frac{1}{2}U = V^{-1}(U)$ belongs to \mathcal{T} . By applying this repeatedly to the set $[0, 1)$ we see that $[0, 2^{-n}) \in \mathcal{T}$ for every $n \in \mathbb{N}$. By applying (a) we now find that $[a, a + 2^{-n}) \in \mathcal{T}$ for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.
- (c) Each set $[a, a + 1)$ belongs to \mathcal{B} , so $\cup \mathcal{B} = \mathbb{R}$. Assume $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. We will show that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. It is easily seen that $B_1 \cap B_2$ is either empty or a set of the form $[a, b)$, with $a < b$. Let $x \in [a, b)$. Then there exists $n \in \mathbb{N}$ such that $x + 2^{-n} < b$. Put $B_3 := [x, x + 2^{-n})$, then it is clear that B_3 satisfies all assertions.
- (d) Let \mathcal{T}_0 be the topology generated by \mathcal{B} . Then $\mathcal{T}_0 \subset \mathcal{T}$. On the other hand, if $B \in \mathcal{B}$ then it is readily verified that $V^{-1}(B) \in \mathcal{B}$ and $T_c^{-1}(B) = B - c \in \mathcal{B}$ for all $c \in \mathbb{R}$. Hence, V and T_c are continuous for \mathcal{T}_0 . Also, $[0, 1) \in \mathcal{T}_0$, so \mathcal{T}_0 satisfies the properties (1), (2), (3). Since \mathcal{T} is the smallest topology with this property, it follows that $\mathcal{T}_0 = \mathcal{T}$.

Solution 2

- (a) Let $x = (z, t) \in S \times J$. The orbit Γx consists of the points (z, t) and $(-z, -t)$. Since $\|z\| = 1$, $z \neq -z$, so (z, t) and $(-z, -t)$ are distinct points.
- (b) Let $\xi \in X/\Gamma$ and let (z, τ) be in the fiber of ξ . By replacing (z, τ) with $(-z, -\tau)$ if necessary, we see that (z, τ) can be found with $z_2 \geq 0$. It follows that there exists $\varphi \in [0, \pi]$ such that $z = (\cos \varphi, \sin \varphi)$. Take $s = \varphi/\pi$ and $t = (\tau + 1)/2$, then $0 \leq s, t \leq 1$ and $\sigma(s, t) = p(z, \tau) = \xi$. Hence σ is surjective.
- (c) Since σ is constant on the classes of \sim there is a unique map $\bar{\sigma} : [0, 1]^2 / \sim \rightarrow X/\Gamma$ such that $\bar{\sigma} \circ q = \sigma$. Clearly, $\bar{\sigma}$ is bijective and continuous. Since $[0, 1]^2$ is compact, and q continuous, $[0, 1]^2 / \sim$ is compact. Since X is Hausdorff, and Γ finite, X/Γ is Hausdorff. It follows that $\bar{\sigma}$ is an embedding. As $\bar{\sigma}$ is surjective, it is a homeomorphism.
- (d) Assertion (1) is incorrect, assertion (2) is correct.

Solution 4

- (a) First assume that f is locally constant and let $z \in Z$. If $y \in f^{-1}(\{z\})$ then there exists an open neighborhood U of y such that f is constant on U hence $f = f(y) = z$ on U , so that $U \subset f^{-1}(\{z\})$. It follows that every point of the fiber $f^{-1}(\{z\})$ is interior, hence the fiber is open.

Conversely, assume every fiber is open, and let $y \in Y$. Put $z = f(y)$ and $U := f^{-1}(\{z\})$. Then U is open and contains y . Furthermore, f is constant on U . We see that f is locally constant.

- (b) First assume that f is locally constant. Let $V \subset Z$ be any subset. Then $f^{-1}(V)$ is the union of the fibers $f^{-1}(\{z\})$, for $z \in V$. All fibers are open by (a), hence $f^{-1}(V)$ is open. It follows that f is continuous.

Conversely, assume that f is continuous for the discrete topology on Z . Then for every $z \in Z$ the set $\{z\}$ is open for the discrete topology, hence $f^{-1}(\{z\})$ is open. It follows from (a) that f is locally constant.

- (c) By (b), we know that f is continuous for the discrete topology on Z . It follows that $f(Y)$ is connected for the induced topology on $f(Y)$, which is the discrete topology. If $z \in f(Y)$ then $\{z\}$ and $f(Y) \setminus \{z\}$ are disjoint open subsets of $f(Y)$, and we see that one of them must be empty. Hence $f(Y) = \{z\}$ and we see that f is constant.

Alternative reasoning: Let Y be connected, and $f : Y \rightarrow Z$ locally constant. If $Y = \emptyset$, there is nothing to prove. Select $y \in Y$ and put $z := f(y)$. Then the fiber $U_1 := f^{-1}(z)$ is open by (a) and contains y hence is not empty. Its complement U_2 in Y equals $f^{-1}(Z \setminus \{z\})$ which is open by (b). From the connectedness of Y we conclude that $U_2 = \emptyset$. Hence $U_1 = Y$ and we see that $f(Y) = \{z\}$, hence, f is constant.

- (d) Assume that Y is not connected. Then $Y = U \cup V$ for certain disjoint non-empty open subsets U and V of Y . We take $Z = \{0, 1\}$ and define $f(y) = 0$ if $y \in U$ and $f(y) = 1$ for $y \in V$. Then clearly, f is locally constant, but not constant.

Solution 3

- (a) For every $a \in X$ there exists an open neighborhood U_a and a constant $m_a > 0$ such that $f_a \geq m_a$ on U_a .

The U_a , for $a \in X$, cover X . By compactness there exists a finite collection a_1, \dots, a_n of points of X such that $X \subset \bigcup_{j=1}^n U_{a_j}$. It follows that $f \geq \min_{1 \leq j \leq n} m_{a_j}$ on each of the sets U_i hence on X . Hence, (a) is valid, with $m_X := \min_{1 \leq j \leq n} m_{a_j}$.

- (b) For every $x \in X$ there exists an open neighborhood U_x of x in X and a constant $m_x > 0$ such that $f \geq m_x$ on U_x . The open sets U_x , for $x \in X$, form an open

covering of X . By paracompactness, this cover has a locally finite refinement, $\{V_i \mid i \in I\}$. By definition of refinement, for every $i \in I$ there exists an $x_i \in X$ such that $V_i \leq U_{x_i}$. It follows that $f \geq m_i := m_{x_i}$ on V_i .

- (c) Let V_i, m_i be as above. Then there exists a partition of unity $\{\eta_i \mid i \in I\}$ which is subordinated to $\{V_i \mid i \in I\}$. Now $0 \leq \eta_i \leq 1$ and $\eta_i = 0$ outside V_i . Therefore, $\eta_i f \geq m_i \eta_i$ on V_i and on $X \setminus V_i$ hence on X . We now note that for every $x \in X$ we have (with finitely many nonzero terms)

$$f(x) = \sum_{i \in I} \eta_i(x) f(x) \geq \sum_{i \in I} m_i \eta_i(x).$$

As the family $\{\eta_i \mid i \in I\}$ is locally finite, the sum

$$\mu := \sum_i m_i \eta_i$$

is a locally finite sum of continuous functions, hence continuous. Since $\sum_i \eta_i = 1$, we have $\mu > 0$ everywhere. Hence μ is a continuous function $X \rightarrow (0, \infty)$ and we have shown that $f \geq \mu$ on X .

Solution 5

- (a) We will show that the complement of $A \times B$ in $X \times Y$ is open. Let $(x, y) \in X \times Y \setminus A \times B$. Then either $x \notin A$ or $y \notin B$. In the first case there exists an open neighborhood U of x in X such that $U \cap A = \emptyset$. Now $U \times Y$ is an open subset of $X \times Y$ which contains (x, y) but has empty intersection with $A \times B$. Treating the second case similarly, we see that for $(x, y) \notin A \times B$ there exists an open neighborhood of (x, y) contained in the complement of $A \times B$. This shows that the complement of $A \times B$ is open.
- (b) Since $\text{cl}(S) \times \text{cl}(T)$ is closed by (a), and contains $S \times T$, it follows that $\text{cl}(S) \times \text{cl}(T) \supset \text{cl}(S \times T)$.
- (c) There exist open $U \subset X$ and $V \subset Y$ such that $(x, y) \in U \times V \subset \mathcal{O}$. Now $U \cap S \neq \emptyset$ and $V \cap T \neq \emptyset$. Since $\mathcal{O} \cap (S \times T) \supset (U \times V) \cap (S \times T) = (U \cap S) \times (V \cap T) \neq \emptyset$, it follows that $\mathcal{O} \cap (S \times T) \neq \emptyset$.
- (d) It follows from (c) that every $(x, y) \in \text{cl}(S) \times \text{cl}(T)$ belongs to the closure of $S \times T$. This shows that $\text{cl}(S) \times \text{cl}(T) \subset \text{cl}(S \times T)$. The converse inclusion has been established in (b).