ENDTERM COMPLEX FUNCTIONS
JUNE 27, 2017, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

**Notation:** For $a \in \mathbb{C}$ and $r > 0$, we write $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, and $\overline{D}(a, r)$ and $C(a, r)$ are the closure and boundary respectively of $D(a, r)$.

**Exercise 1 (10 pt):**
Evaluate the following integral (which clearly is convergent).
\[
\int_0^\infty \frac{1}{(x^2 + 4)(x^2 + 9)} \, dx.
\]

**Exercise 2 (15 pt):**
Fix $R > 0$ and $a \in \mathbb{C}$; we write $D := D(a, R)$ and $\overline{D} := \overline{D}(a, R)$ and $C := C(a, R)$. Let $f, g : \overline{D} \to \mathbb{C}$ be continuous functions, analytic on $D$, such that $|f(z)| = |g(z)|$ for all $z \in C$, and such that $f$ and $g$ have no zeros in $\overline{D}$. Show that $f = \alpha g$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

**Exercise 3 (15 pt):**
Let $a, b \in \mathbb{C}$. Consider the polynomial $p(z) = z^7 + az^4 + bz^2 - 2$.

(a) Show that if $|z| \leq 1/\sqrt{2}$, then
\[
|p(z)| \geq \frac{32 - \sqrt{2} - 4|a| - 8|b|}{16}.
\]

(b) Suppose that $|b| + 3 < |a| \leq \frac{15}{2} - 2|b|$. Show that, counting zeros with their multiplicities, $p$ has
(i) no zeros in the disk $|z| \leq 1/\sqrt{2}$,
(ii) four zeros in the annulus $1/\sqrt{2} < |z| < 1$,
(iii) three zeros in the annulus $1 < |z| < 2$,
(iv) and no zeros in the annulus $2 \leq |z|$. 

Exercise 4 (15 pt): Let
\[ f(z) = \frac{z^2(z-1)e^z}{\sin^2 \pi z} \]
and let \( U \subset \mathbb{C} \) be the domain of \( f \). Let \( V \subset \mathbb{C} \) be the maximal open set on which a holomorphic function \( g \) can be defined that agrees with \( f \) on \( U \). For each \( v \in V \), determine the radius of convergence of the power series for \( g \) at \( v \).

Exercise 5 (15 pt):
Let \( f \) be a non-constant entire function. Prove that the closure of \( f(\mathbb{C}) \) equals \( \mathbb{C} \).

Exercise 6 (20 pt):
Prove that the following integral converges and evaluate it.
\[ \int_0^\infty \frac{\log x}{x^3+1} \, dx. \]
(Hint: Use a contour consisting of two circular arcs and two segments, with ‘vertices’ \( \epsilon, R, Rc, \) and \( \epsilon c \), where \( c^3 = 1, c \neq 1 \). Use the natural substitution to relate the integrals over the two segments. Use an appropriate definition of the complex logarithm.)