Exercise 1

Questions

a) For which $c \in \mathbb{R}$ is $M_c := F^{-1}(c)$ a smooth submanifold of $\mathbb{R}^3$? Give a sketch of $M_c$ for all $c \in \mathbb{R}$.

Solution: The derivative of $F$ is given by $d_{(x,y,z)}F = (2x, 2y, -2z) : \mathbb{R}^3 \to \mathbb{R}$ and is surjective for all $(x, y, z) \neq 0$. So $(0, 0, 0) \notin F^{-1}(0)$ is the only point where $F$ is not a submersion, showing that $M_c = F^{-1}(c)$ is a smooth submanifold of $\mathbb{R}^3$ for all $c \neq 0$. (Sketch: see last page of this file)

$M_0$ is not a smooth submanifold, in fact it is not even a topological manifold! It is given by a cone, and a neighborhood of the tip of the cone can not be homeomorphic to a disc, for it becomes disconnected if you remove the tip itself.

b) For a nonzero vector $v$ in $\Lambda^1$ independent in $\Lambda^k$, show that $\Lambda^1(v)$ is a smooth submanifold of $\Lambda^k$.

Solution: $\Lambda^1 = \{(x, y, z) | x^2 + y^2 = 1 + z^2 \}$. This is a union of circles as $z$ varies. Explicitly, define $\varphi : S^1 \times \mathbb{R} \to M_1$ by $\varphi(a, b, t) := (\sqrt{1 + F^2 a}, \sqrt{1 + t^2 b}, t)$ where $t \in \mathbb{R}$ and $(a, b) \in S^1 \subset \mathbb{R}^2$ (seen as the unit circle). This is smooth map, as it is the restriction of the smooth map from $\mathbb{R} \times S^2 \to \mathbb{R}^3$ given by the same formula. Define $\psi : M_1 \to S^1 \times \mathbb{R}$ by $\psi(x, y, z) := (\frac{1}{\sqrt{1 + x^2 + y^2}}, \frac{1}{\sqrt{1 + z^2}}, y, z)$. This is again smooth map, hence it is the inverse to $\varphi$.

Next consider $M_{-1} = \{(x, y, z) | |x| = 1 + y^2 + 1\} = M_{-1}^+ \bigcup M_{-1}^-$ where $M_{-1}^+ := \{(x, y, z) | z = \pm \sqrt{x^2 + y^2 + 1}\}$. Both $M_{-1}^+$ and $M_{-1}^-$ are graph of a smooth function on $\mathbb{R}^2$, hence are diffeomorphic to $\mathbb{R}^2$. (The diffeomorphism itself can be taken to be the projection $(x, y, z) \mapsto (x, y)$, with inverse $(x, y) \mapsto (x, y, \pm \sqrt{x^2 + y^2 + 1})$.)

Exercise 2

a) Let $V$ and $W$ be vector spaces and $L : V \to W$ a linear map. Recall that the rank of $L$ is the dimension of its image $L(V) \subset W$. Show that the rank of $L$ is the biggest integer $k$ for which $\Lambda^k L : \Lambda^k V \to \Lambda^k W$ is nonzero. (Hint: construct a convenient basis for $V$.)

Solution: Let $v_1, \ldots, v_n$ be a basis for $V$ such that $\ker(L) = \langle v_{k+1}, \ldots, v_n \rangle$. The elements $L v_1, \ldots, L v_k$ then form a basis for $\text{Im}(L)$ and $k$ equals the rank of $L$. Now a basis for $\Lambda^k V$ is given by all the products $v_{i_1} \wedge \ldots \wedge v_{i_l}$ for which $1 \leq i_1 < \ldots < i_l \leq n$. Then,

$$\Lambda^k(L)(v_{i_1} \wedge \ldots \wedge v_{i_k}) = L v_{i_1} \wedge \ldots \wedge L v_{i_k}$$

which clearly is zero if $l > k$, for then necessarily $i_l > k$. Moreover,

$$\Lambda^k(L)(v_{i_1} \wedge \ldots \wedge v_k) = L v_{i_1} \wedge \ldots \wedge L v_k$$

is nonzero because the elements $L v_1, \ldots, L v_k$ are linearly independent.

b) For a nonzero vector $v \in V$ we consider for each $k \geq 0$ the linear map $v \wedge : \Lambda^k V \to \Lambda^{k+1} V$ given by $\alpha \mapsto v \wedge \alpha$. Show that its kernel is given by the image of $v \wedge : \Lambda^{k-1} V \to \Lambda^k V$. (Hint: construct a convenient basis for $V$.)

Solution: Since $v \neq 0$ we can complement it to a basis $v_1, \ldots, v_n \in V$, where $v_1 = v$. For $k = 0$ we know that $v \wedge : \Lambda^0 V = \mathbb{R} \to \Lambda^1 V = V$ is injective, since it maps 1 to $v \neq 0$.

For $k > 0$, the elements $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ lie in the kernel of $v_1 \wedge$, while the elements $v_i \wedge v_{i+1} \wedge \ldots \wedge v_k$ with $1 < i_1 < \ldots < i_k \leq n$ are mapped by $v_1 \wedge$ to a basis of $\text{Im}(v_1 \wedge)$. Indeed, this follows from the fact that the elements $v_1 \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_k$ are all linearly independent in $\Lambda^{k+1} V$. So we see that $\ker(v_1 \wedge) = \langle v_1 \wedge v_{i_2} \wedge \ldots \wedge v_k \ | \ 1 < i_2 \ldots < i_k \leq n \rangle$, which is precisely the image of $v_1 \wedge : \Lambda^{k-1} V \to \Lambda^k V$. 

1
Exercise 3 (30 pt) Consider the two-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$.

a) Compute $\int_{S^2(r)} \omega$, where $S^2(r) := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2 \}$ is the two-sphere of radius $r > 0$ in $\mathbb{R}^3$.

Solution: By Stokes we know that $\int_{S^2(r)} \omega = \int_{B(r)} df = 3 \int_{B(r)} dx \wedge dy \wedge dz$, where $B(r) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq r^2 \}$. Using polar coordinates this gives

$$\int_{S^2(r)} \omega = 4\pi r^3.$$  

b) Let $\alpha := f \cdot \omega \in \Omega^2(\mathbb{R}^3 \backslash 0)$ where $f$ is the function given by $f(x, y, z) := (x^2 + y^2 + z^2)^{-\frac{3}{2}}$.

Show that $d\alpha = 0$ and use this to conclude that $\int_{S^2(r)} \alpha$ is independent of $r \in \mathbb{R}_{>0}$. What is its value?

Solution: We have $d\alpha = df \wedge \omega + f d\omega$, and

$$df \wedge \omega = -\frac{3}{2} (x^2 + y^2 + z^2)^{-5/2} (2xdx + 2ydy + 2zdz) \wedge (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$$

$$= -3f dx \wedge dy \wedge dz = -f d\omega,$$

which shows that $d\alpha = 0$. Now consider the manifold $A(r_1, r_2) := \{(x, y, z) \in \mathbb{R}^3 | r_1 \leq x^2 + y^2 + z^2 \leq r_2 \}$ for $0 < r_1 < r_2$, with boundary given by $S^2(r_2) \backslash S^2(r_1)$. Stokes then gives us

$$0 = \int_{A(r_1, r_2)} df = \int_{S^2(r_2)} \alpha - \int_{S^2(r_1)} \alpha.$$

This shows that the integral is independent of $r \in \mathbb{R}_{>0}$. To see what it is, we take $r = 1$. There $f = 1$, hence $\alpha = \omega$, and from a) we deduce that $\int_{S^2(r)} \alpha = 4\pi$.

c) Let $V$ be the vector field on $\mathbb{R}^3 \backslash 0$ given by $V_{(x,y,z)} := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$. Compute the flow $\varphi^V_t$ of $V$ and show that $(\varphi^V_t)^* \alpha = \alpha$. Use this to give another proof of the fact that $\int_{S^2(r)} \alpha$ is independent of $r$.

Solution: The flow of $V$ is given by $\varphi^V_t(x, y, z) := e^t(x, y, z)$. Indeed, $\varphi^V_0 = \text{Id}$, and

$$\frac{d}{dt} \varphi^V_t(x, y, z) = e^t \varphi^V_t(x, y, z) = V_{e^t(x,y,z)} = V_{\varphi^V_t(x,y,z)}.$$

To show that $(\varphi^V_t)^* \alpha = \alpha$ we can do two things. We can check it directly:

$$(\varphi^V_t)^* \alpha = (\varphi^V_t)^* f \cdot (\varphi^V_t)^* \omega = e^{-3t} f \cdot e^{3t} \omega = f \omega = \alpha$$

or we can compute the Lie derivative:

$$\mathcal{L}_V \alpha = \iota_V d\alpha + d\iota_V \alpha = 0,$$

because $d\alpha = 0$ and $\iota_V \omega = f \iota_V \omega$ and $\iota_V \omega = 0$ as one readily verifies. Since $\varphi^V_t$ gives orientation preserving diffeomorphisms from $S^2(r)$ to $S^2(e^t r)$, we see that $\int_{S^2(r)} (\varphi^V_t)^* \alpha = \int_{S^2(e^t r)} \alpha$. (The fact that $\varphi^V_t$ is orientation preserving follows from its explicit formula, but it is true for every flow in general because it can be continuously deformed to the identity map.)

Exercise 4 (30 pt) For this exercise you may use without proof that $\int_{S^n} : H^n(S^n) \to \mathbb{R}$ is an isomorphism. Let $\pi : S^n \to \mathbb{RP}^n$ denote the quotient map and $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the antipodal map $x \mapsto -x$. 


Exercise 5

a) Show that a form \( \omega \in \Omega^k(S^n) \) is of the form \( \omega = \pi^* \alpha \) for a unique \( \alpha \in \Omega^k(\mathbb{RP}^n) \) if and only if \( \iota^* \omega = \omega \). Deduce that \( \frac{1}{2}(\omega + \iota^* \omega) \in \pi^* (\Omega^k(\mathbb{RP}^n)) \) for every \( \omega \in \Omega^k(S^n) \).

**Solution:** Since \( \pi_* = \pi \) it follows that \( \iota^* \pi^* \alpha = \pi^* \alpha \). In particular, \( \iota^* (\pi^* \alpha) = \pi^* \alpha \) for all \( \alpha \in \Omega^k(\mathbb{RP}^n) \). Conversely, suppose that \( \iota^* \omega = \omega \) for \( \omega \in \Omega^k(S^n) \). The map \( \pi : S^n \to \mathbb{RP}^n \) is a local diffeomorphism, and the inverse image of a point \([x] \in \mathbb{RP}^n\) consists of two points; \( \pm x \in S^n \). Hence, there are neighborhoods \( V \) of \([x] \) in \( \mathbb{RP}^n \) and \( U_\pm \) of \( \pm x \) in \( S^n \) such that \( \pi|_{U_\pm} : U_\pm \to V \) is a diffeomorphism. So, there are unique \( \alpha_\pm \in \Omega^k(V) \) with the property that \( (\pi|_{V\pm})^* \alpha_\pm = \omega|_{U_\pm} \). Observe that \( (\pi|_{U_\pm})^* \alpha_- \circ \iota = (\pi|_{|U_\pm})^* \alpha_+ = \iota^* (\pi|_{U_\pm})^* \alpha_+ = \iota^* \omega|_{U_\pm} = \omega|_{U_\pm} = (\pi|_{U_\pm})^* \alpha_+ \). In particular, \( \alpha_+ = \alpha_- \). We have now shown that around each point in \( \mathbb{RP}^n \) there is a unique \( \alpha \) with the desired property. By uniqueness all these locally constructed \( \alpha \)'s glue together into a globally defined \( \alpha \in \Omega^k(\mathbb{RP}^n) \). The last question follows immediately since \( \iota^* (\frac{1}{2}(\omega + \iota^* \omega)) = \frac{1}{2}(\iota^* \omega + \omega) \), because \( \iota \circ \iota = \text{Id} \).

b) If \( n \) is even and \( \iota^* \omega = \omega \), show that \( \int_{S^n} \omega = 0 \).

**Solution:** Let \( D^+ , D^- \subset S^n \) be the upper and lower hemisphere. Then \( \iota \) induces a diffeomorphism \( \iota : D^- \to D^- \) which for \( n \) even is orientation reversing. Indeed, for \( n \) even the map \( \iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) is orientation reversing, and maps an outward normal of \( S^n \) to another outward normal of \( S^n \). Hence the restriction \( \iota|_{S^n} : S^n \to S^n \) is orientation reversing. Consequently:

\[
\int_{S^n} \omega = \int_{D^+} \omega + \int_{D^-} \omega = \int_{D^+} \omega + \int_{D^-} \iota^* \omega = \int_{D^+} \omega - \int_{D^+} \omega = 0.
\]

c) Show that \( H^n(\mathbb{RP}^n) = 0 \) for all even \( n \). Deduce that \( \mathbb{RP}^n \) is not orientable for \( n \) even.

**Solution:** Let \( \omega \in \Omega^n(\mathbb{RP}^n) \). By a) and b) we know that \( \int_{S^n} \pi^* \omega = 0 \), so by the given fact about \( H^n(S^n) \) we know that \( \pi^* \omega = da \) for some \( a \in \Omega^{n-1}(S^n) \). This \( a \) need not satisfy \( \iota^* a = a \), but we can consider \( \tilde{\alpha} := \frac{1}{2}(a + \iota^* a) \). We have \( \iota^* \tilde{\alpha} = \tilde{\alpha} \), while \( \tilde{\alpha} = \frac{1}{2}(da + \iota^* da) = \pi^* \omega \). By part a) again we can write \( \tilde{\alpha} = \pi^* \beta \) for some \( \beta \in \Omega^{n-1}(\mathbb{RP}^n) \), and we have \( \pi^* \omega = \pi^* d \beta \). Using a) once more, this implies \( \omega = d \beta \), hence \( \omega \) is exact. As \( \omega \) was arbitrary, \( H^n(\mathbb{RP}^n) = 0 \).

---

**Exercise 5** (20 pt) Recall that a vector bundle \( \pi : E \to M \) is called orientable if we can choose an orientation on each fiber, in such a way that around each point in \( M \) we can find a positively oriented frame.

a) Show that a line bundle (i.e. a vector bundle of rank 1) is trivial if and only if it is orientable.

**Solution:** Clearly if \( E \) is trivial, i.e. isomorphic to \( M \times \mathbb{R} \), it is orientable since we can pick a nowhere vanishing section and let that induce an orientation on each fiber. Conversely, suppose that \( E \) is orientable. Choose an open cover \( \{U_a\} \) of \( M \) together with positively oriented sections \( s_a \in \Gamma(E|_{U_a}) \). Let \( \{\rho_a\} \) be a partition of unity subordinate to \( \{U_a\} \). Then \( \rho_a s_a \in \Gamma(E) \), i.e. is \( E \) is a globally defined section of \( E \). Let \( s := \sum_a \rho_a s_a \in \Gamma(E) \) (this is well-defined since the sum is locally finite). If \( x \in M \), \( s(x) = \sum_a \rho_a(x) s_a(x) \), and all the \( s_a(x) \) are nonnegative in \( E_x \) with respect to the given orientation. Moreover, whenever \( \rho_a(x) > 0 \), \( s_a(x) > 0 \), and we know that there is at least one \( a \) for which this is true since \( \sum_a \rho_a = 1 \).

b) Show that for any line bundle \( E \) over \( M \) the line bundle \( E \otimes \mathbb{C} \) is trivial. (Hint: use a))

**Solution:** We will construct an orientation on \( E \otimes \mathbb{C} \). This is based on the following observation: if \( v \in E_x \) is nonzero, it defines a nonzero element \( v \otimes v \in E_x \otimes E_x \), hence an orientation on \( E_x \otimes E_x \). Moreover, if \( w = \lambda v \) is another nonzero element of \( E_x \), then \( w \otimes w = \lambda^2 v \otimes v \). Since \( \lambda^2 > 0 \) for every \( \lambda \neq 0 \), we see that this orientation on \( E_x \otimes E_x \) is independent of
the choice of nonzero vector in $E_x$. We endow all the fibers $E_x \otimes E_x$ of $E \otimes E$ with this orientation. This is an orientation of $E$, i.e. is continuous, because if $e$ is a local frame for $E$, then $e \otimes e$ is a frame for $E \otimes E$ which is positive. Hence, $E \otimes E$ is oriented and so trivial by part a).
\[ C > 0 : \mathcal{M}_C = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + c^2 \} \]

\[ C < 0 : \mathcal{M}_C = \{ (x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 - c \} \]

\[ z = \pm \sqrt{x^2 + y^2 - c} \]