Notes:
1. Write your name and student number **clearly** on each page of written solutions you hand in.
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult textbooks and class notes.
5. You are not allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.
7. Every individual question is worth 10 points, giving a total of 140 points for the entire exam.

Questions

Exercise 1 (30 pt) Consider the map $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $F(x, y, z) := x^2 + y^2 - z^2$.

a) For which $c \in \mathbb{R}$ is $M_c := F^{-1}(c)$ a smooth submanifold of $\mathbb{R}^3$? Give a sketch of $M_c$ for all $c \in \mathbb{R}$.

b) Show that $M_1$ is diffeomorphic to $S^1 \times \mathbb{R}$ and that $M_{-1}$ is diffeomorphic to $\mathbb{R}^2 \coprod \mathbb{R}^2$.

c) Construct an atlas for $M_1$ and compute the transition maps.

Exercise 2 (20 pt)

a) Let $V$ and $W$ be vector spaces and $L : V \rightarrow W$ a linear map. Recall that the rank of $L$ is the dimension of its image $L(V) \subset W$. Show that the rank of $L$ is the biggest number $k$ for which $\Lambda^k L : \Lambda^k V \rightarrow \Lambda^k W$ is nonzero.

b) For a nonzero vector $v \in V$ we consider for each $k \geq 0$ the linear map $v \wedge : \Lambda^k V \rightarrow \Lambda^{k+1} V$ given by $\alpha \mapsto v \wedge \alpha$. Show that its kernel is given by the image of $v \wedge : \Lambda^{k-1} V \rightarrow \Lambda^k V$. (Hint: construct a convenient basis for $V$.)

Exercise 3 (30 pt) Consider the two-form $\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ on $\mathbb{R}^3$.

a) Compute $\int_{S^2(r)} \omega$, where $S^2(r) := \{(x, y, z) | x^2 + y^2 + z^2 = r^2\}$ is the two-sphere of radius $r > 0$ in $\mathbb{R}^3$.

b) Let $\alpha := f \cdot \omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ where $f$ is the function given by $f(x, y, z) := (x^2 + y^2 + z^2)^{-\frac{3}{2}}$. Show that $d\alpha = 0$ and use this to conclude that $\int_{S^2(r)} \alpha$ is independent of $r \in \mathbb{R}_{>0}$. What is its value?

c) Let $V$ be the vector field on $\mathbb{R}^3 \setminus \{0\}$ given by $V_{(x, y, z)} := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$. Compute the flow $\varphi^V_t$ of $V$ and show that $(\varphi^V_t)^* \alpha = \alpha$. Use this to give another proof of the fact that $\int_{S^2(r)} \alpha$ is independent of $r$. 

Exercise 4 (30 pt) For this exercise you may use without proof that $\int_{S^n} : H^n(S^n) \to \mathbb{R}$ is an isomorphism. Let $\pi : S^n \to \mathbb{R}P^n$ denote the quotient map and $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the antipodal map $x \mapsto -x$.

a) Show that a form $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\alpha$ for a unique $\alpha \in \Omega^k(\mathbb{R}P^n)$ if and only if $\iota^*\omega = \omega$. Deduce that $\frac{1}{2}(\omega + \iota^*\omega) \in \pi^*(\Omega^k(\mathbb{R}P^n))$ for every $\omega \in \Omega^k(S^n)$.

b) If $n$ is even and $\iota^*\omega = \omega$, show that $\int_{S^n} \omega = 0$.

c) Show that $H^n(\mathbb{R}P^n) = 0$ for all even $n$. Deduce that $\mathbb{R}P^n$ is not orientable for $n$ even.

(Hint: for $\omega \in \Omega^n(\mathbb{R}P^n)$ show that $\pi^*\omega$ is exact. Then use part a) to write $\pi^*\omega = d\alpha$ for some $\alpha$ with $\iota^*\alpha = \alpha$.)

Exercise 5 (30 pt) Recall that a vector bundle $\pi : E \to M$ is called orientable if we can choose an orientation on each fiber, in such a way that around each point in $M$ we can find a positively oriented frame.

a) Show that a line bundle (i.e. a vector bundle of rank 1) is trivial if and only if it is orientable.

b) Show that for any line bundle $E$ over $M$ the line bundle $E \otimes E$ is trivial.

c) Show that the Möbius bundle over $S^1$ is not trivial.