Differentiable manifolds 2016-2017: Retake

Notes:

1. Write your name and student number **clearly** on each page of written solutions you hand in.

2. You can give solutions in English or Dutch.

3. You are expected to explain your answers.

4. You are allowed to consult text books and class notes.

5. You are not allowed to consult colleagues, calculators, computers etc.

6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

7. Every individual question is worth 10 points, giving a total of 100 points for the entire exam.

Questions

Exercise 1 (20 pt) Consider the manifold $\mathbb{RP}^2$. Using homogeneous coordinates $[x_1 : x_2 : x_3]$ to denote points in $\mathbb{RP}^2$, let $U_i := \{[x_1 : x_2 : x_3] | x_i \neq 0\}$ and $\varphi_i : U_i \rightarrow \mathbb{R}^2$ given by $[x_1 : x_2 : x_3] \mapsto \frac{1}{x_i}(x_j, x_k)$, where $j < k$ are such that $\{i, j, k\} = \{1, 2, 3\}$, denote the standard atlas on $\mathbb{RP}^2$.

a) Prove that this is not an oriented atlas. Does that prove that $\mathbb{RP}^2$ is not orientable? Explain your answer.

b) Consider the polynomial $F(x_1, x_2, x_3) := x_3(x_2)^2 - x_1(x_1 - x_3)(x_1 - \lambda x_3)$ where $\lambda \in \mathbb{R}$. Give equations that describe the zero set $Z(F) := \{[x_1 : x_2 : x_3] \in \mathbb{RP}^2 | F(x_1, x_2, x_3) = 0\}$ in the three charts $U_1$, $U_2$ and $U_3$. Determine those values of $\lambda \in \mathbb{R}$ for which $Z(F)$ is a smooth submanifold of $\mathbb{RP}^2$.

Exercise 2 (20 pt) Let $V$ be a finite dimensional vector space and $W \subset V$ a subspace. Denote by $\text{Ann}(W) := \{\alpha \in V^* | \alpha(w) = 0 \ \forall w \in W\}$ the space of one-forms on $V$ that annihilate $W$.

a) For $k \geq 0$ consider the restriction map $\Lambda^k V^* \rightarrow \Lambda^k W^*$ given by $\alpha \mapsto \alpha|_W$. Show that this is surjective and that the kernel is spanned by elements of the form $\{\alpha_1 \wedge \ldots \wedge \alpha_k | \alpha_1 \in \text{Ann}(W)\}$. (Hint: construct a convenient basis for $V^*$.)

We will denote the kernel of the restriction map $\Lambda^k V^* \rightarrow \Lambda^k W^*$ by $I(k)$.

b) Let $g$ be a positive definite inner product on $V$. Show that, using $g$, there is a natural way (i.e. independent of further choices) of extending elements of $\Lambda^k W^*$ to elements of $\Lambda^k V^*$. Use this do give a decomposition

$$\Lambda^k V^* = \Lambda^k W^* \oplus I(k).$$
Exercise 3(20 pt) Consider the two-form
\[ \omega := \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \]
on \mathbb{R}^3 \setminus 0. Recall from the Final Exam that \( \omega \) is closed and that its integral over the two sphere of radius \( r > 0 \) is independent of \( r \) and equal to \( 4\pi \). You may use these facts without proof in this exercise.

a) Let \( T_1 := \{(x,y,z) | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\} \) denote the two-dimensional torus in \( \mathbb{R}^3 \) that one obtains by rotating the circle \( \{(x,0,z) \in \mathbb{R}^3 | (x-2)^2 + z^2 = 1\} \) around the \( z \)-axis. Compute \( \int_{T_1} \omega \).

b) Let \( T_2 := \{(x,y,z) | (\sqrt{(x-2)^2 + y^2} - 2)^2 + z^2 = 1\} \) denote the two-dimensional torus in \( \mathbb{R}^3 \) that one obtains by translating \( T_1 \) over the \( x \)-axis. Compute \( \int_{T_2} \omega \).

Exercise 4(20 pt) Let \( M = \mathbb{R}^2 \). Consider the maps
\[ H : M \times \mathbb{R} \to M, \quad H(x,y,t) := (e^t x, y) \]
\[ K : M \times \mathbb{R} \to M, \quad K(x,y,t) := (x, y + tx). \]
a) Show that \( H \) and \( K \) are the flows of vector fields \( X_H \), respectively, \( X_K \). Determine \( X_H \) and \( X_K \) explicitly.

b) Let us write \( H_t : M \to M \) for the map \( (x,y) \mapsto H(x,y,t) \), and similarly for \( K \). Show that
\[ \frac{\partial}{\partial t} \Big|_{t=0} \frac{\partial}{\partial s} \Big|_{s=0} K_{-s} H_{-t} K_s H_t (p) = [X_H, X_K](p) \]
for every \( p \in M \).

Exercise 5(20 pt) Let \( M \) be a smooth \( n \)-dimensional manifold and suppose that \( X_1, \ldots, X_n \) are vector fields on \( M \) such that \( X_1(p), \ldots, X_n(p) \) forms a basis of \( T_p M \) for all \( p \in M \). Let \( \alpha^1, \ldots, \alpha^n \) be the one-forms on \( M \) dual to the \( X_i \), determined by the relation \( \alpha^i(X_j) = \delta^i_j \).

a) Show that \([X_i, X_j] = 0\) for all \( i, j \) if and only if \( d\alpha^i = 0 \) for all \( i \).

b) Suppose that \( H^1_{dR}(M) = 0 \). Show that the vector fields \( X_1, \ldots, X_n \) are equal to the coordinate vector fields \( \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \) for some coordinate system \( (x^1, \ldots, x^n) \) on \( M \) if and only if \([X_i, X_j] = 0\).
(Hint: use the assumption together with part a) to find functions \( x^1, \ldots, x^n \) on \( M \) such that \( dx^i(X_j) = \delta^i_j \).)