FINAL EXAM 'INLEIDING IN DE GETALTHEORIE'

Tuesday, 8th November 2016, 8.30 am - 11.30 am

Question 1

a) Find the continued fraction expansion to $\sqrt{28}$.
b) What number has the continued fraction expansion

$$[5, 3, 2, 3, 10, 3, 2, 3, 10, ...] \quad ?$$

Question 2

Let $f(x, y) \in \mathbb{Z}[x, y]$ be a polynomial with integer coefficients in the variables x, y. For a natural number $n \in \mathbb{N}$ we define the function

$$\rho(n) := |\{(x, y) \in (\mathbb{Z}/n\mathbb{Z})^2 : f(x, y) \equiv 0 \bmod n\}|.$$

Here we write $|S|$ for the cardinality of a set S.
(a) Show that $\rho(n)$ is a multiplicative function.
(b) Consider the function $f(x, y) = x^2 - 2y^2$. Give a formula for $\rho(n)$ for squarefree odd positive integers n in terms of Legendre symbols.

Question 3

Let p be an odd prime number and q a prime number which divides $2^p - 1$. Show that $q = 2mp + 1$ for some $m \in \mathbb{N}$.

Question 4

Let p be a prime number with $p \geq 11$. Show that there is an $a \in \{1, 2, \ldots, 9\}$ such that

$$\left(\frac{a}{p}\right) = \left(\frac{a+1}{p}\right) = 1.$$

Question 5

Let $d \in \mathbb{N}$. Find all rational solutions to the equation $x^2 - dy^2 = 1$.

Question 6

Square numbers are numbers of the form n^2 for $n \in \mathbb{N}$. Similarly, we call a number of the form $\frac{3n^2-n}{2}$ with $n \in \mathbb{N}$ a pentagonal number. Find a natural number larger than one which is at the same time a square number and a pentagonal number. Describe a method how one could list all natural numbers which are simultaneously square numbers and pentagonal numbers.

Date: 8th November 2016.

Note: A simple non-programmable calculator is allowed for the exam.