Measure and Integration Solutions Quiz Extra, 2016-17

1. Let $X$ be a set and $\mathcal{F}$ a collection of real valued functions on $X$ satisfying the following properties:

(i) $\mathcal{F}$ contains the constant functions,
(ii) if $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f + g, fg, cf \in \mathcal{F}$,
(iii) if $f_n \in \mathcal{F}$, and $f = \lim_{n \to \infty} f_n$, then $f \in \mathcal{F}$.

For $A \subseteq X$, denote by $1_A$ the indicator function of $A$, i.e.

$$1_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Show that the collection $\mathcal{A} = \{A \subseteq X : 1_A \in \mathcal{F}\}$ is a $\sigma$-algebra. (2.5 pts.)

**Proof** Since $1_X(x) = 1$ for all $x \in X$, then $1_X$ is the constant function 1 so by property (i), $1_X \in \mathcal{F}$ and hence $X \in \mathcal{A}$. Now, let $A \in \mathcal{A}$, then $1_A \in \mathcal{F}$. Since $1_{A^c} = 1 - 1_A$, then by property (ii) we have $1_{A^c} \in \mathcal{F}$ so $A^c \in \mathcal{A}$. Finally, consider a sequence $(A_n)$ with $A_n \in \mathcal{A}$, then $1_{A_n} \in \mathcal{F}$ for all $n$, and by the above $1_{A_n} \in \mathcal{F}$ for all $n$. By property (ii), we have $1_{A_1}1_{A_2} \cdots 1_{A_n} \in \mathcal{F}$, hence $1_{\bigcup_{n=1}^{\infty} A_n} = 1 - 1_{A_1^c}1_{A_2^c} \cdots 1_{A_n^c} \in \mathcal{F}$ for all $n$. Since $1_{\bigcup_{n=1}^{\infty} A_n} = \lim_{n \to \infty} 1_{\bigcup_{m=1}^{n} A_m}$, then by property (iii) $1_{\bigcup_{n=1}^{\infty} A_n} \in \mathcal{F}$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Thus $\mathcal{A}$ is a $\sigma$-algebra.

2. Let $(X, \mathcal{D}, \mu)$ be a measure space, and let $\overline{\mathcal{D}}'$ be the completion of the $\sigma$-algebra $\mathcal{D}$ with respect to the measure $\mu$ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension of the measure $\mu$ to the $\sigma$-algebra $\overline{\mathcal{D}}'$. Suppose $f : X \to X$ is a function such that $f^{-1}(B) \in \mathcal{D}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{D}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}'$ and $\overline{\mu}(f^{-1}(B)) = \overline{\mu}(B)$ for all $B \in \overline{\mathcal{D}}'$. (2.5 pts.)

**Proof**: Let $B \in \overline{\mathcal{D}}'$, then there exist $A, B \in \mathcal{D}$ such that $A \subseteq B \subseteq B$, $\mu(B \setminus A) = 0$ and $\overline{\mu}(B) = \mu(A)$. Then, $f^{-1}(A), f^{-1}(B) \in \mathcal{D}$ satisfy $f^{-1}(A) \subseteq f^{-1}(B) \subseteq f^{-1}(B)$ and $\mu(f^{-1}(B) \setminus f^{-1}(A)) = \mu(f^{-1}(B) \setminus A) = 0$. Thus, $f^{-1}(B) \in \overline{\mathcal{D}}'$ and $\overline{\mu}(f^{-1}(B)) = \mu(f^{-1}(A) = \mu(A) = \overline{\mu}(B)$.

3. Consider the measure space $([0,1]|\mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel $\sigma$-algebra to $[0,1]$, and $\lambda$ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1, \cdots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least $n$ sets in the collection $\{E_j\}_{j=1}^{m}$, where $n \leq m$. Show that there exists a $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$. (2.5 pts.)
Proof: By hypothesis, for any \( x \in [0, 1] \) we have \( \sum_{j=1}^{m} 1_{E_j}(x) \geq n \). Assume for the sake of getting a contradiction that \( \lambda(E_j) < \frac{n}{m} \) for all \( 1 \leq j \leq m \). Then,

\[
n = \int_{[0,1]} n \, d\lambda \leq \sum_{j=1}^{m} \lambda(E_j) = \sum_{j=1}^{m} \frac{n}{m} = n,
\]
a contradiction. Hence, there exists \( j \in \{1, \ldots, m\} \) such that \( \lambda(E_j) \geq \frac{n}{m} \).

4. Let \( \mu \) and \( \nu \) be two measures on the measure space \( (E, \mathcal{B}) \) such that \( \mu(A) \leq \nu(A) \) for all \( A \in \mathcal{B} \). Show that if \( f \) is any non-negative measurable function on \( (E, \mathcal{B}) \), then \( \int_E f \, d\mu \leq \int_E f \, d\nu \). (2.5 pts.)

Proof Suppose first that \( f = 1_A \) is the indicator function of some set \( A \in \mathcal{B} \). Then

\[
\int_E f \, d\mu = \mu(A) \leq \nu(A) = \int_E f \, d\nu.
\]

Suppose now that \( f = \sum_{k=1}^{n} \alpha_k 1_{A_k} \) is a non-negative measurable simple function. Then,

\[
\int_E f \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \leq \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f \, d\nu.
\]

Finally, let \( f \) be a non-negative measurable function, then there exists a sequence of non-negative measurable simple functions \( f_n \) such that \( f_n \uparrow f \). By Beppo-Levi,

\[
\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu \leq \lim_{n \to \infty} \int_E f_n \, d\nu = \int_E f \, d\nu.
\]