Exercise 1. [Coupon bond] Consider a coupon bond with face value $F$ and maturity equal to $N$ years, paying a coupon $C$ at the end of each year. The effective yearly interest rate is $r$.

(a) (0.5 pts.) Show that the price of such bond is

$$V_0 = \frac{C}{r} \left[1 - \left(\frac{1}{1+r}\right)^N\right] + \frac{F}{(1+r)^N}.$$ 

(b) (0.5 pts.) An investor purchases the bond but decides to sell it immediately after having received the $k$-th coupon. Find the selling price.

Exercise 2. [Replication with selling fee and interest spread] Two scenarios are foreseen for a certain stock after one period: one in which the stock value is $110$ E and another in which the value is $90$ E. Its current value is $S_0 = 100$ E. Furthermore:

- Each operation of selling the stock to the market carries a fee of 2% (there is no fee to buy from the market).
- Borrowing money costs 12% and deposits pay only 8%.

A call option is established at a strike price also equal to $100$ E. Determine:

(a) (0.8 pts.) The risk-neutral probability.

(b) (0.8 pts.) The fair price of the option.

(c) (0.8 pts.) The hedging strategy.

Exercise 3. [Filtrations and (non-)stopping times] Two numbers are randomly generated by a computer. The only possible outcomes are the numbers 1, 2 or 3. The corresponding sample space is $\Omega_2 = \{(\omega_1, \omega_2) : \omega_i \in \{1, 2, 3\}\}$. Consider the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$, where $\mathcal{F}_0$ is formed only by the empty set and $\Omega_2$, $\mathcal{F}_1$ formed by all events depending only on the first number, and $\mathcal{F}_2$ all events in $\Omega_2$ (this is the ternary version of the two-period binary scenario discussed in class).

(a) (0.8 pts.) List all the events forming $\mathcal{F}_1$.

(b) (0.8 pts.) Let $\tau : \Omega_2 \rightarrow \mathbb{N} \cup \{\infty\}$ defined as the “last outcome equal to 3”. That is, $\tau(3, \omega_2) = 1$ if $\omega_2 \neq 3$, $\tau(\omega_1, 3) = 2$ for all $\omega_1$, and $\tau = \infty$ if no 3 shows up. Prove that $\tau$ is not a stopping time with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$. 


Exercise 4. [Put options] Consider a stock with initial price $S_0$ following a binomial model with $u = 2$ and $d = 1/2$. That is, at the end of each period, the price can either double or be halved. Bank interest is 25% for each period. A producer will have the stock available at the end of two periods and wishes to sell it for at least $S_0$ at that time.

(a) (2pts.) The producer is offered three possibilities:

(O1) A forward selling contract
(O2) An European put option
(O3) An American put option with intrinsic value $G(S) = S_0 - S$.

Compute the fair initial price of each of the possibilities.

(b) The investor purchases the American option.

-i- (1 pt.) Establish the optimal exercise time $\tau^*$ for the investor.

-ii- (1 pt.) Verify the validity of the formula

$$\text{Value of the American option} = \mathbb{E}\left[1_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1 + r)^{\tau^*}}\right].$$

-iii- (1 pt.) Show that the discounted values $\tilde{V}_n$ do not form a martingale, but the stopped discounted values $\tilde{V}_{\tau^*}^n$ do.

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Bonus problem

Bonus. [Dividend-paying stock] (2 pts.) Consider the general binary (not necessarily binomial) dividend-paying stock model. The model is defined by stock prices $S_n$ and growth factors $R_n$, $n = 0, \ldots, N$. At the end of each period, after the new stock value is attained, a dividend is paid and the stock price is reduced by the corresponding amount Formally, these operations are described by the following adapted non-negative random variables

(a) $Y_n(\omega_1, \ldots, \omega_n)$ representing the percentual change in stock value from time $t_{n-1}$ to $t_n$, that is, before paying dividend at $t_n$. Hence, the stock value at $t_n$ is

$$S_n^- = Y_n S_{n-1}.$$

(b) $A_n(\omega_1, \ldots, \omega_n)$ representing the percent of the $t_n^-$-value of the stock paid as a dividend at $t_n^+$. Thus,

$$S_n = (1 - A_n) Y_n S_{n-1}.$$

If the financial institution adopts hedging strategies $\Delta_n$, the wealth equation for the values $X_n$ of its portfolio becomes

$$X_{n+1} = \Delta_n Y_{n+1} S_n + R_n (X_n - \Delta_n S_n).$$

Consider the risk-neutral measure defined by (omitting, as done in class, the overall dependence on $\omega_1, \ldots, \omega_n$)

$$\tilde{p}_n = \frac{R_n S_n - S_{n+1}^- (T)}{S_{n+1}^- (H) - S_{n+1}^- (T)} = \frac{R_n - Y_{n+1}^- (T)}{Y_{n+1}^+ (H) - Y_{n+1}^- (T)}.$$

Show the following:
(a) (0.5 pts) $\tilde{E}(Y_{n+1} \mid \mathcal{F}_n) = R_n$.

(b) (0.5 pts) The discounted wealth process $\bar{X}_n$ is a $\tilde{P}$-martingale, whichever the hedging strategy.

(c) (0.5 pts) The discounted stock price $\bar{S}_n$ is not a $\tilde{P}$-martingale, but only a $\tilde{P}$-super-martingale.

(d) (0.5 pts) In contrast, the process

$$\hat{S}_n = \frac{\bar{S}_n}{(1 - A_1) \cdots (1 - A_n)}$$

is a martingale.