Utrecht University
Introductory mathematics for finance
WISB373
Winter 2016

Exam July 6, 2016

JUSTIFY YOUR ANSWERS
Allowed material: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE: The test consists of four problems plus a bonus problem for a total of 12 points. The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Coupon bond] Consider a coupon bond with face value F and maturity equal to N years, paying a coupon C at the end of each year. The effective yearly interest rate is r.

(a) (0.5 pts.) Show that the price of such bond is

$$V_0 \;=\; \frac{C}{r}\Big[1-\Big(\frac{1}{1+r}\Big)^N\Big]+\frac{F}{(1+r)^N}\;.$$

(b) (0.5 pts.) An investor purchases the bond but decides to sell it immediately after having received the k-th coupon. Find the selling price.

Exercise 2. [Replication with selling fee and interest spread] Two scenarios are foreseen for a certain stock after one period: one in which the stock value is 110 E and another in which the value is 90E. Its current value is $S_0 = 100$E. Furthermore:

- Each operation of selling the stock to the market carries a fee of 2% (there is no fee to buy from the market).
- Borrowing money costs 12% and deposits pay only 8%.

A call option is established at a strike price also equal to 100E. Determine:

(a) (0.8 pts.) The risk-neutral probability.

(b) (0.8 pts.) The fair price of the option.

(c) (0.8 pts.) The hedging strategy.

Exercise 3. [Filtrations and (non-)stopping times] Two numbers are randomly generated by a computer. The only possible outcomes are the numbers 1, 2 or 3. The corresponding sample space is $\Omega_2 = \big\{(\omega_1,\omega_2) : \omega_i \in \{1,2,3\}\big\}$. Consider the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$, where $\mathcal{F}_0$ is formed only by the empty set and Ω_2, $\mathcal{F}_1$ formed by all events depending only on the first number, and $\mathcal{F}_2$ all events in Ω_2 (this is the ternary version of the two-period binary scenario discussed in class).

(a) (0.8 pts.) List all the events forming $\mathcal{F}_1$.

(b) (0.8 pts.) Let $\tau : \Omega_2 \longrightarrow \mathbb{N}\cup\{\infty\}$ defined as the "last outcome equal to 3". That is, $\tau(3,\omega_2) = 1$ if $\omega_2 \neq 3$, $\tau(\omega_1, 3) = 2$ for all ω_1, and $\tau = \infty$ if no 3 shows up. Prove that τ is *not* a stopping time with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$.

Exercise 4. [Put options] Consider a stock with initial price S_0 following a binomial model with $u = 2$ and $d = 1/2$. That is, at the end of each period, the price can either double or be halved. Bank interest is 25% for each period. A producer will have the stock available at the end of two periods and wishes to sell it for at least S_0 at that time.

(a) (2pts.) The producer is offered three possibilities:

(O1) A forward selling contract

(O2) An European put option

(O3) An American put option with intrinsic value $G(S) = S_0 - S$.

Compute the fair initial price of each of the possibilities.

(b) The investor purchases the American option.

-i- (1 pt.) Establish the optimal exercise time τ^* for the investor.

-ii- (1 pt.) Verify the validity of the formula

$$\text{Value of the American option} \;=\; \widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau^*\le N\}}\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\Big]\,.$$

-iii- (1 pt.) Show that the discounted values $\overline{V}_n$ do *not* form a martingale, but the stopped discounted values $\overline{V_n^{\tau^*}}$ do.

Bonus problem

Bonus. **[Dividend-paying stock]** (2 pts.) Consider the general binary (not necessarily binomial) dividend-paying stock model. The model is defined by stock prices S_n and growth factors R_n, $n = 0, \ldots, N$. At the end of each period, after the new stock value is attained, a dividend is paid and the stock price is reduced by the corresponding amount Formally, these operations are described by the following adapted non-negative random variables

(a) $Y_n(\omega_1, \ldots, \omega_n)$ representing the percentual change in stock value from time t_{n-1}^+ to t_n^-, that is, *before* paying dividend at t_n. Hence, the stock value at t_n^- is

$$S_n^- \;=\; Y_n S_{n-1}\;.$$

(b) $A_n(\omega_1, \ldots, \omega_n)$ representing the percent of the t_n^--value of the stock paid as a dividend at t_n^+. Thus,

$$S_n \;=\; (1 - A_n) Y_n S_{n-1}\;.$$

If the financial institution adopts hedging strategies Δ_n, the wealth equation for the values X_n of its portfolio becomes

$$X_{n+1} \;=\; \Delta_n Y_{n+1} S_n + R_n (X_n - \Delta_n S_n)\;.$$

Consider the risk-neutral measure defined by (omitting, as done in class, the overall dependence on $\omega_1, \ldots, \omega_n$)

$$\widetilde{p}_n \;=\; \frac{R_n S_n - S_{n+1}^-(T)}{S_{n+1}^-(H) - S_{n+1}^-(T)} \;=\; \frac{R_n - Y_{n+1}(T)}{Y_{n+1}(H) - Y_{n+1}(T)}\;.$$

Show the following:

(a) (0.5 pts) $\widetilde{E}(Y_{n+1} \mid \mathcal{F}_n) \;=\; R_n$.

(b) (0.5 pts) The discounted wealth process $\overline{X}_n$ is a $\widetilde{P}$-martingale, whichever the hedging strategy.

(c) (0.5 pts) The discounted stock price $\overline{S}_n$ is *not* a $\widetilde{P}$-martingale, but only a $\widetilde{P}$-super-martingale.

(d) (0.5 pts) In contrast, the process

$$\widehat{S}_n \;=\; \frac{\overline{S}_n}{(1 - A_1) \cdots (1 - A_n)}$$

is a martingale.