JUSTIFY YOUR ANSWERS
Allowed: calculator, material handed out in class and handwritten notes (your handwriting). NO BOOK IS ALLOWED

NOTE:
• The test consists of five exercises for a total of 10 credits plus two bonus problems for a maximum of 1.5 pts.
• The score is computed by adding all the valid credits up to a maximum of 10.

Exercise 1. (0.6 pts.) Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with mean $\mu$ and variance $\sigma^2$ and let $N$ be an integer-valued random variable of mean $\lambda$ independent of the previous ones. Define

$$S = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i.$$ 

Determine $E(S^2)$ as a function of $\mu$, $\sigma^2$ and $\lambda$.

Exercise 2. Consider a Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1/3 & 0 & 1/3 & 1/3
\end{pmatrix}$$

(a) (0.4 pts.) Show that $P^n_{44} = (1/3)^n$.

(b) (0.6 pts.) Show that the state “4” is transient.

(c) (0.6 pts.) Let $T = \inf\{n > 0 : X_n \neq 4\}$ be the time it takes the process to exit “4” (for ever). Compute $E(T \mid X_0 = 4)$. \textit{[Hint:} you may want to use that for a discrete random variable $Z$, $E[Z] = \sum_{k=0}^{\infty} P(Z > k).$]\textit{]}

(d) (0.6 pts.) Let $T_3 = \inf\{n > 0 : X_n = 3\}$ be the absorption time at state “3”. Compute $P(T = T_3 \mid X_0 = 4)$, that is the probability that the process exist “4” only to be absorbed by “3”.

(e) (0.6 pts.) Compute all the invariant measures of the process.

Exercise 3. Let $X_1, X_2$ and $X_3$ be independent exponential random variables with respective rates $\lambda_1$, $\lambda_2$ and $\lambda_3$. Compute:

(a) (0.6 pts.) $P(X_1 > X_2 + t \mid X_1 > t)$.

(b) (0.6 pts.) $P(X_1 > X_2 + t \mid X_2 > t)$. 

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Exercise 4. Let \( \{N(t) : t \geq 0\} \) be a Poisson process with rate \( \lambda \). Find

(a) (0.6 pts.) \( P(N(5) = 5, N(17) = 17, N(20) = 20) \).

(b) (0.6 pts.) \( E[N(20) \mid N(17) = 17, N(5) = 5] \).

(c) (0.6 pts.) Determine \( \lambda \) such that
\[
E[N(20) \mid N(17) = 17] = E[N(17) \mid N(20) = 40].
\]

Exercise 5. An atom subjected to electromagnetic radiation oscillates between its ground state \( G \) and two excited states \( E_1 \) and \( E_2 \). Measurements show that the time the atom remains in each excited state is exponentially distributed with mean \( 1/4 \) (picoseconds), after which the atom relaxes to the ground state. Once relaxed, the atom remains in the ground state an exponential time with mean \( 1 \). Due to its lower energy, the atom goes 3 times more often to the state \( E_1 \) than to \( E_2 \).

(a) (0.6 pts.) Model this evolution as a continuous-time Markov chain among the positions \( E_1, G, E_2 \). That is, determine the abandoning rates \( \nu_G, \nu_{E_1} \) and \( \nu_{E_2} \) and the transitions \( P_{ij} \) with \( i,j = E_1, G, E_2 \).

(b) (0.4 pts.) Can this process be interpreted as a birth-and-death process?

(c) (0.6 pts.) Determine, in the long run, the fraction of time spent by the atom in each of the three positions.

(d) (0.8 pts.) Write the 9 backward Kolmogorov equations, and observe that they form three sets of three coupled linear differential equations.

(e) (0.6 pts.) Determine \( P_{E_1 E_1}(t) - P_{E_2 E_1}(t) \).

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Bonus problems

Only one of them may count for the grade
You can try both, but only the one with the highest grade will be considered

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**Bonus 1. [Not all states can be transient]** Consider a homogeneous (or shift-invariant) Markov chain \( (X_n)_{n \in \mathbb{N}} \) with finite state space \( S \). Let us recall that the hitting time of a state \( y \) is
\[
T_y = \min\{n \geq 1 : X_n = y\}.
\]

(a) If \( \ell \leq n \in \mathbb{N}, x,y \in S \), prove the following

-\( \text{i-} \) (0.5 pt.)
\[
P(X_n = y, T_y = \ell \mid X_0 = x) = P_{yy}^{n-\ell} P(T_y = \ell \mid X_0 = x).
\]

-\( \text{ii-} \) (0.5 pt.)
\[
P_{xy}^n = \sum_{\ell=1}^{n} P_{yy}^{n-\ell} P(T_y = \ell \mid X_0 = x).
\]
(b) Conclude the following:

- i- (0.3 pt.) If every state is transient, then for every $x, y \in S$,

$$\sum_{n \geq 0} P_{xy}^n < \infty .$$

- ii- (0.2 pt.) The previous result leads to a contradiction with the stochasticity property of the matrix $P$. Hence not all states can be transient.

**Bonus 2. [Invariant probabilities are indeed invariant]** Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with countable state-space $S = \{x_1, x_2, \ldots\}$, waiting rates $\nu_i$ and embedded transition matrix $P_{ij}$, $i, j \geq 1$. Let $(P_i)_{i \geq 1}$ be an invariant probability distribution, that is, a family of positive numbers $P_i$ satisfying $\sum_i P_i = 1$ and

$$\sum_{k : k \neq i} P_k \nu_k P_{ki} = \nu_i P_i$$

for all $i \geq 1$. Prove that if the process is initially distributed with the invariant law $(P_i)$, this law is kept for the rest of the evolution. That is, prove that

$$P(X(0) = x_i) = P_i \implies P(X(t) = x_i) = P_i$$

for all $t \geq 0$. **Suggestion:** Follow the following steps.

(i) (0.5 pt.) Show that if $P(X(0) = x_i) = P_i$, then

$$P(X(t) = x_j) = \sum_i P_i P_{ij}(t) .$$

(ii) (0.7 pt.) Use Kolmogorov backward equations to show that, as a consequence,

$$\frac{d}{dt} P(X(t) = x_j) = 0$$

for all $t \geq 0$.

(iii) (0.3 pt.) Conclude.