Final exam, Mathematical Modelling (WISB357)

Wednesday, 1 February 2017, 9.00-12.00, BBG 161

- Write your name on each page you turn in, and additionally, on the first page, write your student number and the total number of pages submitted.
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- For this midterm exam you are allowed to bring an A4 with notes on one side. You may not consult solutions to the problems, nor use a graphical calculator or smart phone.

Solution. In small type-font letters.
Scoring.
Maximum possible points per part is shown in the margin.
Your score is the total number of points received divided by 3.

Problem 1. This problem concerns the “reaction-diffusion” equation

\[ u_t = D u_{xx} - cu, \quad \text{for} \quad \begin{cases} -\infty < x < \infty, \\ 0 < t, \end{cases} \]

with the initial condition

\[ u(x,0) = f(x). \]

Assume \( c \) and \( D \) are positive constants.

(a) Using the Fourier Transform, find the solution of the above problem.

5 Solution. Let \( U_k(t) = \mathcal{F}u(x,t), \ k \in \mathbb{Z} \) be the Fourier transform of \( u(x,t) \) in space, and \( \mathcal{F}_k(f(x)) \). Then transforming the partial differential equation and initial condition yields

\[ \dot{U}_k = -(k^2D + c)U_k, \quad U_k(0) = F_k, \quad k \in \mathbb{Z}. \]

The exact solution is

\[
U_k(t) = \exp \left[ -(k^2D + c)t \right] F_k = e^{-ct} \left[ e^{-k^2D t} F_k \right]
\]

Applying the inverse transform from Table 4.1 of the textbook to the terms in brackets we find the solution

\[ u(x,t) = \frac{e^{-ct}}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} \exp \left[ \frac{(x-s)^2}{4Dt} \right] f(s) \, ds. \]

(b) Show that the problem can also be solved by applying the transformation \( u = ve^{at} \), for a carefully chosen constant \( a \), followed by using known solution of the diffusion equation.
Solution. Let \( u(x,t) = v(x,t)e^{at} \), hence
\[
\begin{align*}
  u_t &= v_t e^{at} + av e^{at}, \\
  u_{xx} &= v_{xx} e^{at}, \\
  u(0,t) &= v(x,0) = f(x).
\end{align*}
\]
Substituting these into the partial differential equation and initial condition yields
\[
\begin{align*}
  v_t e^{at} + av e^{at} &= Dv_{xx} e^{at} - cv e^{at}, \\
  v(x,0) &= f(x)
\end{align*}
\]
Choosing \( a = -c \) eliminates the second term. Dividing by \( e^{at} \) leaves the diffusion equation
\[
\begin{align*}
  v_t &= Dv_{xx}, \\
  v(x,0) &= f(x),
\end{align*}
\]
for which the solution is
\[
v(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} \exp\left[ \frac{(x-s)^2}{4Dt} \right] f(s) \, ds.
\]
To get \( u(x,t) \) we just multiply this by \( e^{at} = e^{-ct} \).

Problem 2. Suppose “traffic” is governed by the Burgers equation
\[
\rho_t + \rho \rho_x = 0,
\]
with initial condition
\[
\rho(x,0) = \begin{cases} 
  0, & x \leq -1, \\
  \frac{1}{2}(1+x), & -1 < x < 1, \\
  1, & 1 \leq x.
\end{cases}
\]
(a) Sketch the characteristics in the \((x,t)\)-plane.

Solution. The equation for constant \( \rho \) is
\[
\rho_t + dX dt \rho_x = 0.
\]
Evidently, \( \frac{dX}{dt} = \rho(X(0),0) \). That is, the characteristic emanating from \( X(0) = x \) has slope \( \rho(x,0) \). The characteristics emanating from \( x < -1 \) are vertical lines, perpendicular to the \( x \)-axis. Those emanating from \( x > 1 \) are lines with slope 1 to the right. The characteristics between -1 and 1 vary in angle from 90° to 45°, with angle decreasing linearly from left to right.

(b) Find the solution, \( \rho(x,t) \), using the method of characteristics.

Solution. The characteristic passing through a point \((x_1,t_1)\) is a line satisfying \( x_1 = x_0 + t_1 \rho(x_0,0) \). Along this line, the density is constant: \( \rho(x_1,t_1) = \rho(x_0,0) \). For \( x_0 < -1 \), we find \( x_1 = x_0 \), hence \( \rho(x_1,t_1) = 0 \).

For \( x_0 > 1 \), we find \( x_1 = x_0 + t_1 \) and \( \rho = 1 \).

For \(-1 \leq x_0 \leq 1 \) we find \( x_1 = x_0 + \frac{t_1}{2}(1+x_0) \). Solving this expression for \( x_0 \) yields
\[
\begin{align*}
  x_0 &= \frac{x_1 - \frac{1}{2}}{1 + \frac{1}{2}}, \\
  \rho(x_1,t_1) &= \rho(x_0,0) = \frac{1 + x_1}{2 + t_1}
\end{align*}
\]
with \( \rho(x_1,t_1) = \frac{1}{2}(x_0 + 1) \). Summarizing, the solution is
\[
\rho(x,t) = \begin{cases} 
  0, & x < -1, \\
  \frac{1 + x}{2 + t}, & -1 \leq x \leq 1 + t, \\
  1, & x > 1 + t.
\end{cases}
\]
(c) Find the points in the \((x, t)\)-plane where \(\rho = 1/3\).

2 Solution. In particular along this characteristic, \(\rho(x_0, 0) = 1/3\). This relation can be solved for \(x_0\) to obtain

\[
\rho(x_0, 0) = \frac{1}{3} = \frac{1}{2}(1 + x_0) \implies x_0 = -\frac{1}{3}
\]

The equation for the characteristic is

\[
x = x_0 + \rho(x_0, 0)t = -\frac{1}{3} + \frac{1}{3}t.
\]

Along this line \(\rho\) is constant and equal to 1/3.

(d) Show that \(v = \frac{1}{2}\rho\). Determine the flux \(J\).

1 Solution. The transport equation is

\[
\rho_t + J(\rho)_x = 0 = \rho_t + J'(\rho)\rho_x.
\]

Evidently, \(J'(\rho) = \rho\), and thus

\[
J(\rho) = \frac{\rho^2}{2}.
\]

Since \(J(\rho) = \rho v(\rho)\), it follows that \(v(\rho) = \rho/2\).

**Problem 3.** A linearly elastic bar is made of two different materials, and before being stretched it occupies the interval \(0 \leq A \leq \ell_0\). Also, before being stretched, for \(0 \leq A < A_0\), the modulus and density are \(E = E_L\) and \(R = R_L\), while for \(A_0 < A < \ell_0\) they are \(E = E_R\) and \(R = R_R\). Both \(R_L\) and \(R_R\) are constants. (Hint: It is useful to define separate functions \(U_L(A), U_R(A), T_L(A), T_R(A)\), etc. on the left and right parts of the domain.)

(a) The requirements at the interface, where \(A = A_0\), are that the displacement and stress are continuous. Express these requirements mathematically, using one-sided limits.

2 Solution. Denote the displacement and stress functions by \(U_L(A)\) and \(T_L(A)\) for \(A < A_0\) and \(U_R(A)\) and \(T_R(A)\) for \(A > A_0\). The boundary conditions are:

\[
\lim_{A \to A_0^-} U_L(A) = \lim_{A \to A_0^+} U_R(A), \quad \lim_{A \to A_0^-} T_L(A) = \lim_{A \to A_0^+} T_R(A).
\]

(b) Suppose the bar is stretched and the boundary conditions are \(U(0, t) = 0\) and \(U(\ell_0, t) = \ell - \ell_0\). Assume there are no body forces. Find the steady state solution for the density, displacement and stress.
Solution. The steady state elastic problem is

\[ 0 = \partial A T(A) = \partial A (E \partial A U(A)), \quad U(0) = 0, \quad U(\ell_0) = \ell - \ell_0. \]

This equation holds on each interval \(0 \leq A \leq A_0\) and \(A_0 \leq A \leq \ell_0\), with the additional boundary conditions at \(A_0\). Consequently, we obtain the following system of differential equations:

\[
\begin{align*}
E_L \frac{\partial^2 U_L}{\partial A^2} &= 0, & U_L(0) &= 0, & U_L(A_0) &= U_R(A_0), \\
E_R \frac{\partial^2 U_R}{\partial A^2} &= 0, & U_R(\ell_0) &= \ell - \ell_0, & E_L U_L'(A_0) &= E_R U_R'(A_0).
\end{align*}
\]

Both displacements are linear functions in \(A\):

\[ U_L(A) = \alpha_L A + \beta_L, \quad U_R(A) = \alpha_R A + \beta_R. \]

The boundary conditions at \(A = 0\) and \(A = \ell_0\) imply

\[ U_L(0) = 0 = \beta_L, \quad U_R(\ell_0) = \ell - \ell_0 = \alpha_R \ell_0 + \beta_R, \]

from which it follows that \(\beta_R = \ell - (1 + \alpha_R) \ell_0\). The conditions at the interface become

\[
\begin{align*}
U_L(A_0) &= U_R(A_0) \iff \alpha_L A_0 &= \alpha_R A_0 + \ell - (1 + \alpha_R) \ell_0, \\
T_L(A_0) &= T_R(A_0) \iff E_L \alpha_L &= E_R \alpha_R.
\end{align*}
\]

This yields a linear system of equations for \(\alpha_L\) and \(\alpha_R\) that can be solved by substitution to find

\[
\begin{align*}
\alpha_L &= \frac{E_R (\ell - \ell_0)}{A_0 (E_R - E_L) + E_L \ell_0}, \\
\alpha_R &= \frac{E_L (\ell - \ell_0)}{A_0 (E_R - E_L) + E_L \ell_0},
\end{align*}
\]

from which the solutions are defined:

\[
\begin{align*}
U(A) &= \begin{cases} 
\alpha_L A, & A \leq A_0, \\
\alpha_R (A - \ell_0) + \ell - \ell_0, & A > A_0,
\end{cases} \\
T(A) &= E_L \alpha_L = E_R \alpha_R, \\
R(A) &= \begin{cases} 
\frac{R_L}{1 + \sigma_L}, & A < A_0, \\
R(A) = \frac{R_R}{1 + \sigma_R}, & A > A_0.
\end{cases}
\end{align*}
\]