1. Let $X = \{X_1, \ldots, X_n\}$ be a random sample of $n$ i.i.d. Poisson random variables with parameter $\lambda$.

   (a) (8pt) Find the maximum likelihood for $\lambda$ and its asymptotic sampling distribution.

   (b) (8pt) Find the maximum likelihood estimator for the parameter $\mu = e^{-\lambda}$.

Suppose now that, rather than observing the actual values of the random variables $X_i$, we are just able to register whether they are null or positive. More precisely, only the events $X_i = 0$ or $X_i > 0$ for $i = 1, \ldots, n$ are observed.

   (c) (8pt) Find the maximum likelihood for $\lambda$ for these new observations.

   (d) (8pt) When does the maximum likelihood estimator not exist? Assuming that the true value of $\lambda$ is $\lambda_0$, compute the probability that the maximum likelihood estimator does not exist.

2. Let $X = \{X_1, \ldots, X_n\}$ be a random sample of $n$ i.i.d. random variables with densities:

$$f_X(x; \theta) = \begin{cases} \frac{\theta^3}{2} x^2 e^{-\theta x} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

with $\theta > 0$ is an unknown parameter. Moreover, consider another random sample $Y = \{Y_1, \ldots, Y_n\}$ of $n$ i.i.d. random variables with densities:

$$f_Y(y; \mu) = \begin{cases} \frac{\mu^3}{2} y^2 e^{-\mu y} & \text{if } y > 0, \\ 0 & \text{otherwise} \end{cases}$$

with $\mu > 0$ is another unknown parameter. We further assume that the two sample are independent (i.e. $X_i \perp Y_j$, for all $i, j$).

   (a) [10pt] Find the Generalized Likelihood Ratio Test (GLRT) statistic for testing:

$$H_0 : \theta = \mu, \quad H_1 : \theta \neq \mu.$$

Let us define now the following statistic:

$$T := \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{j=1}^n Y_j}$$

   (b) [10pt] Show that the GLRT rejects $H_0$ if $T(1 - T) < k$, for a suitable constant $k$. 
3. A company wants to monitor the efficiency of two employees in completing an assigned task. For this reason, the performances of two employees (denoted by A and B) were measured by recording the times needed to complete the assigned tasks. Hence, the following two samples have been collected:

\[ x_A = \{5.18, 13.43, 6.31, 3.18, 4.91, 11.07\}, \]
\[ x_B = \{5.50, 18.16, 8.14, 9.14, 14.24, 10.72\} \]

where the duration of each task is measured in hours.

(a) [10pt] Perform a test at 10% of significance for testing the hypothesis that employee A is faster than B. Discuss critically the choice of the test used.

Suppose now that the time \( T \) needed by an employee for completing a task can be modeled by a continuous random variable with the following probability density function:

\[
f_T(t; \theta) = \begin{cases} \frac{1}{2\sqrt{\theta}} e^{-\frac{t^2}{4\theta}} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases}
\]

(1)

with \( \theta > 0 \) an unknown parameter.

(b) [8pt] Given a sample \( T = \{T_1, \ldots, T_n\} \) of i.i.d random variables sampled from \( f_T(t; \theta) \), determine the maximum likelihood estimator of the probability \( P_\theta(T > 7) \).

(c) [8pt] Under the parametric model (1) for the random variable \( T \) and given the samples \( x_A, x_B \), estimate the probability that the time needed by an employee for completing a task is larger than 7 hours, under the further assumption that 55% of the employees are similar to employee A and 45% to employee B.

4. Let the independent random variables \( Y_1, Y_2, \ldots, Y_n \) be such that we have the following linear model:

\[ Y_i = \beta_0 + \beta_1 x_i + \beta_2 (x_i - 3.5)_+ + \epsilon_i \]

for \( i = 1, \ldots, n \), where \( \epsilon_i \) are i.i.d. normal random variables such that \( \epsilon_i \sim N(0, \sigma^2) \) and with \( (y)_+ \) we denoted the positive part of the real number \( y \) (i.e. \( (y)_+ := \max(0, y) \)). We collect the following sample of observations:

\[ y = \{1, 2, 4, 5, 4, 3, 1\} \]

corresponding to the predictors:

\[ x = \{0, 1, 2, 3, 4, 5, 6\} \]

(a) [8pt] If we rewrite the linear model using the usual matrix formalism

\[ Y = X\beta + \epsilon \]

write down the design matrix \( X \) of the linear model.

(b) [6pt] Given that

\[
(X^\top X)^{-1} = \begin{pmatrix} 0.65 & -0.24 & 0.35 \\ -0.24 & 0.14 & -0.26 \\ 0.35 & -0.26 & 0.65 \end{pmatrix}
\]

estimate the model coefficients and write down the fitted model.

(b) [8pt] Calculate the prediction of the fitted model at \( x = 4.5 \). Assuming that the sum of squared residuals equals 7.8, calculate a 95% confidence interval for this prediction.