Exam Inleiding Topologie, 30/1-2017, 13:30 - 16:30

Solution 1.

(a) Let \( a < b, a' < b' \) and \( x \in \mathbb{R} \) be real numbers such that \( x \in [a, b) \cap [a', b') \). Then 
\[ d'' := \max(a, a') \leq x \text{ and } b'' := \min(b, b') > x. \] It follows that \( x \in [d'', b'') \subset [a, b) \cap [a', b') \). This establishes the assertion.

(b) It is straightforward to see that \( T \) is a bijection with inverse \( T^{-1} : y \mapsto p^{-1}y - q/p \). Thus we see that the pre-image of an interval of the form \( [a, b) \) equals 
\[ T^{-1}([a, b)) = [a', b'), \]
with \( a' = a/p - q/p \) and \( b' = b/p - q/p \). Thus, \( T^{-1}([a, b)) \in \mathcal{T} \). Since the sets \([a, b)\) form a basis of \( \mathcal{T} \) we see that \( T \) is continuous. Since \( T^{-1} \) is of similar type, we see that \( T^{-1} \) is continuous as well. Hence, \( T \) is a homeomorphism.

(c) We first observe that 
\[ (0, 1) = \bigcup_{n \geq 1} \left[ \frac{1}{n}, 1 \right). \]
Thus, \( (0, 1) \) is a union of sets from \( \mathcal{T} \). By applying item (b), we find that every set of the form \( (q, q + p) \) with \( p, q \in \mathbb{R} \) and \( p > 0 \) belongs to \( \mathcal{T} \). Since the sets \((q, q + p)\) form a basis of the topology for \( \mathbb{R}^{eucl} \), the inclusion follows.

(d) Let \( x, y \in \mathbb{R}, x \neq y \). Since \((\mathbb{R}, \mathcal{T}_{eucl})\) is (metrizable hence) Hausdorff, there exist \( U, V \in \mathcal{T}_{eucl} \) such that \( U \ni x, V \ni y \) and \( U \cap V = \emptyset \). By (c) we have \( U, V \in \mathcal{T} \).

Hence, \((\mathbb{R}, \mathcal{T})\) is Hausdorff.

(e) The identity map \( I : \mathbb{R} \to \mathbb{R} \) is continuous \((\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{eucl})\) and maps \( S \) to \( S \). Thus, if \( S \) is compact in \((\mathbb{R}, \mathcal{T})\) then its image \( S \) under \( I \) is compact in \((\mathbb{R}, \mathcal{T}_{eucl})\).

Alternative solution: Assume that \( S \) is compact with respect to \( \mathcal{T} \). Let \( \{U_i\}_{i \in I} \) be an open cover of \( S \) with sets from \( \mathcal{T}_{eucl} \). By the previous item, each set \( U_i \) belongs to \( \mathcal{T} \), so that \( \{U_i\}_{i \in I} \) is an open cover of \( S \) relative to \( \mathcal{T} \). Since \( S \) is compact relative to \( \mathcal{T} \) the cover contains a finite subcover. Hence, \( S \) is compact relative to \( \mathcal{T}_{eucl} \).

(f) We observe that \([a, \infty) = \bigcup_{n \geq 1} [a, n) \) belongs to \( \mathcal{T} \) hence its complement \((-\infty, a) \) is closed in \((\mathbb{R}, \mathcal{T})\) and it follows that \( S \cap (-\infty, a) \) is closed in \( S \), relative to (the restriction of) \( \mathcal{T} \). Since \( S \) is compact for \( \mathcal{T} \), it follows that \( S \cap (-\infty, a) \) is compact for \( \mathcal{T} \).

(g) The set \([0, 1) = [0, 1] \cap (-\infty, 1) \) is closed in \([0, 1] \), relative to the topology induced by \( \mathcal{T} \), by item (f). If \([0, 1] \) were compact for \( \mathcal{T} \), then \([0, 1) = [0, 1] \cap (-\infty, 1) \) would be compact for \( \mathcal{T} \) by hence also for \( \mathcal{T}_{eucl} \), by (e). This is a contradiction, since all compact subsets of \((\mathbb{R}, \mathcal{T}_{eucl})\) are closed in \((\mathbb{R}, \mathcal{T}_{eucl})\). It follows that \([0, 1] \) is not compact for \( \mathcal{T} \).
(h) Assume \( (\mathbb{R}, \mathcal{J}) \) were locally compact. Then there would be a compact neighborhood \( N \) of 0 relative to \( \mathcal{J} \). Now \( N \) would contain a set of the form \([0, 2\delta) \in \mathcal{J} \), for \( \delta > 0 \). Hence \( N \subseteq [0, \delta] \). The set \([0, \delta] \) is closed in \((\mathbb{R}, \mathcal{J}_{eucl})\) hence in \((\mathbb{R}, \mathcal{J})\), by (c). It follows that \([0, \delta] \) is closed in \( N \) relative to the restriction of \( \mathcal{J} \), hence compact. This contradicts the conclusion of the previous item, in view of (b).

Solution 2.

(a) By definition, \( Y \) is the collection of sets \( \Gamma x \), for \( x \in \mathbb{R} \). Furthermore, \( \pi : \mathbb{R} \to Y \) is given by \( \pi(x) = \Gamma x \). Now \( \Gamma \cdot 0 = \{0\} \), \( \Gamma \cdot (-1) = (-\infty, 0) \) and \( \Gamma \cdot 1 = (0, \infty) \). The unit of these sets is \( \mathbb{R} \). Thus, we see that \( \mathbb{R} \) splits into 3 \( \Gamma \)-orbits, namely the ones containing \(-1, 0, 1\). These orbits are precisely the points \( a, b \) and \( c \) in \( Y \).

(b) A set \( S \subseteq Y \) is open for the quotient topology if and only if \( \pi^{-1}(S) \) is open. Now \( \pi(S) \) is the union of the fibers \( \pi^{-1}(y) \), for \( y \in Y \). The fibers are: \( \pi^{-1}(a) = \Gamma \cdot (-1) = (-\infty, 0) \) \( \pi^{-1}(b) = \Gamma \cdot 0 = \{0\} \) and \( \pi^{-1}(c) = \Gamma \cdot 1 = (0, \infty) \). From this we see that 
\[
\mathcal{Y} \supseteq \{\theta, Y, \{a\}, \{c\}, \{a, c\}\}.
\]

If \( U \in \mathcal{Y} \) contains \( b \), then \( \pi^{-1}(U) \) must contain 0. For it to be a union of the fibers and open in \( \mathbb{R} \), it needs to contain \( \mathbb{R} \). Hence, \( U = Y \). It follows that the inclusion \( \supseteq \) is an equality.

(c) The space \( Y \) is not Hausdorff. Indeed, the only set from \( \mathcal{Y} \) containing \( b \) is \( Y \). Thus, every neighborhood of \( b \) contains \( Y \) and we see that this topology is not Hausdorff.

By definition the map \( \pi \) is continuous. Since \( \mathbb{R} \) is connected, and \( \pi \) surjective, it follows that \( Y \) is connected.

Alternative approach: One may use the description under (b) as follows. Let \( U, V \in \mathcal{Y} \) and assume \( Y = U \cup V, U \cap V = \emptyset \). Without loss of generality we may assume that \( b \in U \). Then \( U = Y \) which forces \( V = 0 \). Hence, \( Y \) is connected for the quotient topology.

Since \( Y \) is finite, every open cover of \( Y \) is already finite, hence \( Y \) is compact.

Solution 3.

(a) Assume (1). Then without loss of generality we may assume that \( X_1 \) is compact. Since \( X^+ \) is Hausdorff, \( X_1 \) is closed in \( X^+ \). Thus, \( X^+ \setminus X_1 \) is open in \( X^+ \) and contains \( X_2 \) hence is non-empty. Also, \( X_1 \) is open in \( X^+ \) and non-empty. We find that \( X^+ \) is the disjoint union of two open non-empty subsets \( X_1 \) and \( X^+ \setminus X_1 \), hence not-connected.

(b) It follows from the assumption that \( U \cap X_j \) is both open and closed in \( X_j \). As \( U \) is the union of these intersections, one of them is non-empty. Without loss of generality we may assume that \( U \cap X_1 \neq \emptyset \). Now \( X_1 \) is the disjoint union of the open subsets \( U \cap X_1 \) and \( X_1 \setminus (U \cap X_1) \). By connectedness of \( X_1 \), the second set must be empty, hence \( U \cap X_1 = X_1 \), so that \( X_1 \subset U \). 

2
(c) Assume (2). Then there exist non-empty open sets $U, V \subset X^+$ which are disjoint and such that $U \cup V = X^+$. As $U, V$ are each other’s complement, they are closed in $X^+$ as well. Hence they are also compact.

Without loss of generality we may assume that $\infty \in V$ so that $U = X^+ \setminus V$ is a subset of $X$. Since the topology on $X$ is induced by the topology on $X^+$, it follows that $U$ is open, closed and compact in $X$. By item (b) we may assume that $X_1$ is contained in $U$. Since $U$ is compact and $X_1$ closed in $U$ it follows that $X_1$ is compact.

(d) Let $X := (-2, -1) \cup (0, 1)$, equipped with the restriction topology of the Euclidean topology on $\mathbb{R}$. Since $X$ is the disjoint union of two non-empty open subsets, it is not connected. Thus $X_1 = (-2, -1)$ and $X_2 = (0, 1)$ are as in the above, and non-compact. It follows that $X^+$ is connected.

**Solution 4.**

(a) Since $X$ is a subspace of a Hausdorff space, it is Hausdorff. As $X$ is the union of the two closed and bounded subsets $D \times \{-1\}$ and $D \times \{+1\}$, the set $X$ is closed and bounded in $\mathbb{R}^3$, hence compact.

(b) We note that $\|\varphi(x, \pm 1)\|^2 = \|x\|^2 + (1 - \|x\|^2) = 1$, hence $\varphi$ maps into the unit sphere. If $y$ is a point of the unit sphere, we may write $y = (x, t)$, with $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$ and then $\|x\|^2 + t^2 = \|y\|^2 = 1$ so that $\|x\|^2 \leq 1$ and $t^2 = (1 - \|x\|^2)$. It follows that $x \in D$ and $t = \pm \sqrt{1 - \|x\|^2}$. Hence $y = \varphi(x, \pm 1)$. This shows that $\varphi$ is surjective.

(c) If $f$ and $g$ belong to $A$, then $(f + g)(x, -1) = f(x, -1) + g(x, -1) = f(x, 1) + g(x, 1) = (fg)(x, 1)$ for all $x \in \partial D$. Hence $f + g \in A$. Similarly one shows that $fg \in A$. If $\lambda \in \mathbb{R}$ and $f \in A$ then for $x \in \partial D$ we have $\lambda f(x, -1) = \lambda f(x, 1) = \lambda f(x, 1)$ and we see that $\lambda f \in A$. Finally, the constant function $1 \in A$ belongs to $A$. It follows that $A$ is a unital subalgebra.

(d) We will determine the fibers $\varphi^{-1}(y)$ of the map $\varphi$. First, let $y = (x, t)$ be a point of the unit sphere with $t \neq 0$. Then it follows from the reasoning in (b) that $(x, \text{sign}(t), 1)$ is the unique element in the fiber $\varphi^{-1}(y)$. Next, let $y = (x, t)$ be in the unit sphere and assume that $t = 0$. Then it follows that $\|x\| = 1$ and $t = 0$, and we see that $\varphi(x', \eta) = (x, 0)$ if and only if $x' = x$ and $\eta \in \{-1, 1\}$, hence $\varphi^{-1}(y)$ consists of the points $(x, \pm 1)$.

It follows from the above that $A$ is precisely the algebra of continuous functions $f : X \to \mathbb{R}$ which are constant on the fibers of $\varphi$. It follows that $\varphi^* : f \mapsto f \circ \varphi$ is a bijection from $C(S^2)$ onto $A$. This bijection is an isomorphism of algebras. Thus, the algebras $A$ and $C(S^2)$ are isomorphic and from this we infer that the topological spectrum $X_A$ is homeomorphic to the topological spectrum of $C(S^2)$. By the Gelfand-Naimark theorem, the latter is homeomorphic to $S^2$. Thus, $X_A$ is homeomorphic to $S^2$. 

3