Retake Inleiding Topologie, 18/4-2017, 13:30 - 16:30

Solution 1.

(a) The sets 0 and \( \mathbb{R} \) belong to \( \mathcal{T} \). If \( U, V \in \mathcal{T} \) then \( \mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V) \) is either \( \mathbb{R} \) or finite, hence \( U \cap V \) belongs to \( \mathcal{T} \). If \( \{ U_i \mid i \in I \} \) is a collection of sets in \( \mathcal{T} \), then the union \( U = \bigcup_{i \in I} U_i \) has complement \( \mathbb{R} \setminus U = \bigcap_{i \in I} (\mathbb{R} \setminus U_i) \). If all sets \( U_i \) are empty, then so is \( U \) hence \( U \) is open in that case. In the remaining case, at least one of \( \mathbb{R} \setminus U_i \) is finite, hence \( \mathbb{R} \setminus U \) is finite and we conclude that \( U \in \mathcal{T} \).

(b) Let \( U_0 \) and \( U_1 \) be any open sets with \( U_0 \supseteq 0 \) and \( U_1 \supseteq 1 \). Then it follows that the set \( \mathbb{R} \setminus (U_0 \cap U_1) = (\mathbb{R} \setminus U_1) \cup (\mathbb{R} \setminus U_2) \) is finite, hence its complement \( U_1 \cap U_2 \) is non-empty. Hence 0 and 1 cannot be separated and we see that topology is not Hausdorff.

(c) The closed sets of \( \mathbb{R} \) with respect to \( \mathcal{T} \) are precisely the finite sets, and \( \mathbb{R} \). Thus, the only closed set contained \( \mathbb{Z} \) is \( \mathbb{R} \), and it follows that the closure of \( \mathbb{Z} \) is \( \mathbb{R} \).

(d) Let \( U \subseteq [0, 1] \) be a set of \( \mathcal{T} \). Then the complement of \( U \) is infinite, hence \( U = \emptyset \). We conclude that the only set of \( \mathcal{T} \) contained in \( [0, 1] \) is the empty set. Hence the interior of \( [0, 1] \) is empty.

(e) Let \( S \) be a subset of \( \mathbb{R} \) and let \( \{ S_i \mid i \in I \} \) be an open cover of \( S \) for the induced topology. Then every \( S_i \) is of the form \( S \cap U_i \), where \( S_i \in \mathcal{T} \). If \( S \) is the emptyset, there is nothing to prove. Thus, assume \( S \) contains a point \( x \). Then \( x \in U_{i_0} \) for some \( i_0 \). It follows that \( \mathbb{R} \setminus U_{i_0} \) is finite, so \( S \setminus S_{i_0} = S \setminus U_{i_0} \) is finite hence consists of elements \( x_1, \ldots, x_N \). For each \( 1 \leq k \leq N \) chose \( i_k \in I \) such that \( x_k \in S_{i_k} \). Then the sets \( S_{i_0}, \ldots, S_{i_N} \) cover \( S \). It follows that \( S \) is compact.

(f) Assume that \( A \) is not connected. Then \( A = A_1 \cup A_2 \), with \( A_1 \) and \( A_2 \) disjoint non-empty open subsets of \( A \) for the induced topology. Then there exist open \( U_j \) of \( X \) such that \( A_j = A \cap U_j \). Clearly, \( U_j \) is non-empty, hence \( \mathbb{R} \setminus U_j \) is finite. It follows that \( A_1 = A \setminus A_2 = A \setminus U_2 \subset \mathbb{R} \setminus U_2 \) hence \( A_1 \) is finite. Likewise, \( A_2 \) is finite. It follows that \( A = A_1 \cup A_2 \) is finite.

Conversely, assume that \( A \) is finite. Select \( a \in A \) and write \( A_1 = \{ a \} \) and \( A_2 = A \setminus A_1 \). Then \( A_2 \) is finite in \( \mathbb{R} \) hence closed. Hence \( A_1 \) and \( A_2 \) are two closed subsets of \( A \) whose disjoint union is \( A \). It follows that \( A_1 \) and \( A_2 \) are open in \( A \) as well, hence \( A \) is not connected.

Solution 2.

(a) Let \( x \in X \). Then there exists an open neighborhood \( U_x \ni x \) such that \( f \vert_{U_x} \) is injective. It follows that for every \( y \in Y \) there can be at most one \( x' \in U_x \) such that \( f(x') = y \). Hence, \( f^{-1}(\{y\}) \cap U_x \) has at most one element. Thus, \( \{ U_x \mid x \in X \} \) is an open covering of \( X \) as asserted.
(b) Let \( \{ U_i \mid i \in I \} \) be a covering as mentioned in (a). Then there exist finitely many indices \( i_1, \ldots, i_N \) such that \( U_{i_1}, \ldots, U_{i_N} \) cover \( X \). Let \( y \in Y \). Then

\[
f^{-1}(y) = f^{-1}(y) \cap (U_{i_1} \cup \cdots \cup U_{i_N}) \subset \bigcup_{k=1}^{N} (f^{-1}(y) \cap U_{i_k}).
\]

In view of (a), this implies that \( \# f^{-1}(\{y\}) \leq N \).

Solution 3.

(a) Let \( i \in I \). Then \( A \cap U_i \) is open and closed in \( U_i \) for the induced topology. Its complement in \( U_i \) is \( U_i \setminus A \), and is closed and open in \( U_i \). It follows that \( U_i \) is the disjoint union of the open subsets \( U_i \cap A \) and \( U_i \setminus A \). One of these sets must be empty since \( U_i \) is connected. If the second set is empty, then \( U_i \subseteq A \) hence \( U_i \cap A = U_i \). The assertion follows.

(b) Assume that \( A \) is not disjoint from \( U_i \). Then it follows from (a) that \( A \) contains \( U_i \). It follows that \( A \) contains \( U_i \cap U_j \) hence is not disjoint from \( U_j \). Again by (a) it follows that \( A \) contains both \( U_i \) and \( U_j \).

Likewise, if \( A \cap U_j \neq \emptyset \) then \( A \) contains both \( U_j \) and \( U_i \). The result follows.

(c) Let \( i \sim j \). There exists a sequence \( i_0, \ldots, i_n \) in \( I \) such that \( i_0 = i \), \( i_n = j \) and \( U_{i_{k-1}} \cap U_{i_k} \neq \emptyset \) for all \( 1 \leq k \leq n \). Assume \( A \cap U_i \neq \emptyset \). Then it follows by applying (b) repeatedly that \( U_{i_k} \subseteq A \) for all \( 0 \leq k \leq n \). In particular, both \( U_i \) and \( U_j \) are contained in \( A \). Likewise, if \( U_j \cap A \neq \emptyset \), then \( A \) contains both \( U_j \) and \( U_i \).

(d) Assume that \( X \) is not connected. Then \( X \) can be written as the disjoint union of two non-empty open sets \( A_1 \) and \( A_2 \). Then \( A_1 \) is open and closed in \( X \) and non-empty. Since the \( U_i \) cover \( X \) it follows that there exists \( i \in I \) such that \( U_i \cap A_1 \neq \emptyset \). Since \( A_1 \) is a proper subset of \( X \) there must be \( j \) such that \( A_1 \not\supset U_j \). By (c) it follows that \( j \) is not equivalent to \( i \). The assertion now follows by contraposition.

(e) Arguing by contraposition, assume that not all elements of \( I \) are equivalent. Let \( i_1 \in I \) be such that \( U_i \neq \emptyset \) and let \( I_1 \) be the equivalence class of \( i_1 \). Let \( A_1 \) be the union of the sets \( U_i \) for \( i \in I_1 \). Then \( A_1 \) is open. If \( j \not\sim i_1 \) then it follows that \( U_j \cap U_i = \emptyset \) for all \( i \in I_1 \) hence \( U_j \cap A_1 = \emptyset \). Thus the union \( A_2 \) of sets \( U_j \) for \( j \in I \setminus I_1 \) is non-empty, open and disjoint from \( A_1 \). Obviously \( A_1 \cup A_2 = X \). It follows that \( X \) is not connected.

Solution 4.

(a) Let \( w, z \in \overline{D} \) be distinct and assume that \( zRw \). Then \( \varphi(z) = \varphi(w) \). Hence, \( z^2 = w^2 \), and we find \( -z = w \), in particular \( |z| = |w| \) and \( z \neq -z \). By looking at the first components of \( \varphi(z) \) and \( \varphi(w) \) we see that \( (1 - |z|)z = (1 - |w|)w = -(1 - |z|)z \). Hence \( (1 - |z|)2z = 0 \) and we see that \( |z| = 1 \). Thus, if \( z, w \in \overline{D} \) are distinct then \( zRw \) implies \( |z| = 1 \) and \( w = -z \).
Conversely, assume that \(|z| = 1\) and \(z = -w\). Then it readily follows that \(z \neq w\) and \(\phi(z) = \phi(w)\). Thus, we see that for different \(z, w \in \bar{D}\) we have \(z R w\) if and only if \(z \in \partial \bar{D}\) and \(w = -z\).

It follows from this that \(\bar{D}/R\) equipped with the quotient topology is homeomorphic to \(\mathbb{P}^2(\mathbb{R})\).

(b) The map \(\phi : \bar{D} \to \mathbb{C}^2\) is continuous, hence factors through an injective continuous map \(\bar{\phi} : \bar{D}/R \to \mathbb{C}^2\). Since \(\bar{D}\) is compact, so is its continuous image \(\bar{D}/R\) and since \(\mathbb{C}^2\) is Hausdorff the map \(\bar{\phi}\) is a topological embedding. Since \(\bar{D}/R\) is homeomorphic to \(\mathbb{P}^2(\mathbb{R})\) and \(\mathbb{C}^2\) is homeomorphic to \(\mathbb{R}^4\), it follows that there exists a topological embedding of \(\mathbb{P}^2(\mathbb{R})\) into \(\mathbb{R}^4\).

(c) Let \(p : \bar{D} \to \bar{D}/R\) be the natural projection. Then the map \(p^* : C(\bar{D}/R) \to C(\bar{D}), f \mapsto f \circ p\) is an injective homomorphism of algebras with image \(A\). It follows that the algebra \(A\) is isomorphic with the algebra \(C(\bar{D}/R)\). It follows that the topological spectrum \(X_A\) is homeomorphic to the topological spectrum of \(C(\bar{D}/R)\) which in turn is homeomorphic to \(\bar{D}/R \simeq \mathbb{P}^2(\mathbb{R})\).

Solution 5.

(a) Assume that \(\hat{f} : \hat{X} \to \hat{Y}\) is continuous. Let \(K \subset Y\) be compact. Then \(V := \hat{Y} \setminus K\) is open in \(\hat{Y}\) hence its preimage \(U := f^{-1}(V)\) is open in \(\hat{X}\). Since \(V\) contains \(\infty_Y\), the open set \(U\) contains \(f^{-1}(\infty_Y) = \infty_X\), hence its complement \(\hat{X}\setminus U\) is closed hence compact, and contained in \(X\). We now note that \(f^{-1}(K) = \hat{f}^{-1}(K) = \hat{f}^{-1}(\hat{Y}\setminus V) = \hat{X}\setminus U\) is compact in \(X\).

(b) Assume that for every compact \(K \subset Y\) the preimage \(f^{-1}(K)\) in \(X\) is compact for the relative topology. Let \(V \subset \hat{Y}\) be an open subset.

Case 1: \(V\) does not contain \(\infty_Y\). Then \(V\) is contained in \(\hat{Y}\) hence \(\hat{f}^{-1}(V)\) equals \(f^{-1}(V)\) hence is open in \(\hat{X}\) by continuity of \(f\). Since \(X\) is open in \(\hat{X}\) it follows that \(f^{-1}(V)\) is open in \(\hat{X}\).

Case 2: \(V \ni \infty_Y\). In this case \(K := \hat{Y}\setminus V\) is closed in \(\hat{Y}\) hence compact. Furthermore, \(K\) is contained in \(\hat{Y}\) hence \(f^{-1}(K)\) is a compact subset of \(X\). It follows that \(\hat{f}^{-1}(K) = f^{-1}(K)\) is compact in \(X\) hence in \(\hat{X}\). Since the latter is Hausdorff, \(\hat{f}^{-1}(K)\) is closed in \(\hat{X}\) and we find that \(\hat{f}^{-1}(V) = \hat{f}^{-1}(\hat{Y}\setminus K) = \hat{X}\setminus \hat{f}^{-1}(K)\) is open in \(\hat{X}\).

It follows that in all cases \(\hat{f}^{-1}(V)\) is open in \(\hat{X}\). Hence, \(\hat{f}\) is continuous and the converse implication has been established.