(1) Let \( \{W(t) : t \geq 0\} \) be a Brownian motion with filtration \( \{\mathcal{F}(t) : t \geq 0\} \). Consider the process \( \{S(t) : t \geq 0\} \) defined by

\[
S(t) = -\int_0^t 2S(u) du + \int_0^t e^{-4u} dW(u).
\]

(a) Show that the process \( \{e^{2t}S(t) : t \geq 0\} \) is a martingale with respect to the filtration \( \{\mathcal{F}(t) : t \geq 0\} \). (1 pt)

(b) Determine the distribution of \( S(t) \). (1 pt)

**Proof (a):** First observe that

\[
S(0) = 0.
\]

We apply Itô product rule, we get

\[
d\left(e^{2t}S(t)\right) = 2e^{2t}S(t) dt + e^{2t} dS(t)
\]

\[
= 2e^{2t}S(t) dt + e^{2t} \left(-2S(t) dt + e^{-4t} dW(t)\right)
\]

\[
= e^{-4t} dW(t).
\]

Since \( S(0) = 0 \), we see that \( e^{2t}S(t) = \int_0^t e^{-2u} dW(u) \) is an Itô-integral and hence the process \( \{e^{2t}S(t) : t \geq 0\} \) is a martingale with respect to the filtration \( \{\mathcal{F}(t) : t \geq 0\} \).

**Proof (b):** From part (a), we have \( S(t) = \int_0^t e^{-2(t+u)} dW(u) \). Note that \( \int_0^t e^{-2(t+u)} dW(u) \) is an Itô-integral of a deterministic process, hence it is normally distributed with mean 0 and variance

\[
\text{Var}\left(\int_0^t e^{-2(t+u)} dW(u)\right) = e^{-4t} \int_0^t e^{-4u} du = \frac{e^{-4t}(1 - e^{-4t})}{4}.
\]

(2) Let \( \{(W_1(t), W_2(t)) : t \geq 0\} \) be a 2-dimensional Brownian motion defined on a probability space \( (\Omega, \mathcal{F}, P) \). Consider the price process \( \{S(t) : t \geq 0\} \) given by

\[
S(t) = 1 + \int_0^t \alpha S(u) dW_1(u) + \int_0^t \beta S(u) dW_2(u)
\]

where \( \alpha, \beta \) are positive constants.

(a) Show that \( \{S^2(t) : t \geq 0\} \) is a 2-dimensional Itô-process. (1pt)

(b) Show that \( \mathbb{E}[S^2(t)] = e^{(\alpha^2 + \beta^2)t} \), \( t \geq 0 \). (You are allowed to interchange integrals and expectations but justify why). (1.5 pts)

**Proof (a):** From the hypothesis, we have \( d(S(t)) = \alpha S(t) dW_1(t) + \beta S(t) dW_2(t) \). Consider the continuously differentiable function \( f(x) = x^2 \). We have \( f_x(x) = 2x, f_{xx}(x) = 2 \) and \( S^2(t) = f(S(t)) \). By Itô-Doeblin,

\[
d\left(S^2(t)\right) = 2S(t) dS(t) + \frac{1}{2} \cdot 2dS(t) dS(t)
\]

\[
= 2S(t) \left(\alpha S(t) dW_1(t) + \beta S(t) dW_2(t)\right) + (\alpha^2 + \beta^2)S^2(t) dt
\]

\[
= (\alpha^2 + \beta^2)S^2(t) dt + 2\alpha S^2(t) dW_1(t) + 2\beta S^2(t) dW_2(t).
\]
Since $S(0) = 1$, we have $S^2(0) = 1$ and $S^2(t) = 1 + \int_0^t (\alpha^2 + \beta^2) S^2(u) du + \int_0^t 2\alpha S^2(u) dW_1(u) + \int_0^t 2\beta S^2(u) dW_2(u)$. Hence $\{S^2(t) : t \geq 0\}$ is a 2-dimensional Itô-process.

**Proof (b):** From part (a),

$$S^2(t) = 1 + \int_0^t (\alpha^2 + \beta^2) S^2(u) du + \int_0^t 2\alpha S^2(u) dW_1(u) + \int_0^t 2\beta S^2(u) dW_2(u).$$

Since Itô-integrals have zero expectation, we see by Fubini’s Theorem that

$$\mathbb{E}[S^2(t)] = 1 + \mathbb{E} \left[ \int_0^t (\alpha^2 + \beta^2) S^2(u) du \right] = 1 + \int_0^t (\alpha^2 + \beta^2) \mathbb{E}[S^2(u)] du.$$

Let $m(t) = \mathbb{E}[S^2(t)]$, then the above equation reads

$$m(t) = 1 + \int_0^t (\alpha^2 + \beta^2) m(u) du,$$

or equivalently $\frac{dm(t)}{dt} = (\alpha^2 + \beta^2) m(t)$ with $m(0) = 1$. This differential equation has solution $m(t) = Ce^{(\alpha^2 + \beta^2)t}$. Since $m(0) = 1$, we have $C = 1$ and hence $m(t) = \mathbb{E}[S^2(t)] = e^{(\alpha^2 + \beta^2)t}$.

(3) Let $T$ be a finite horizon and let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, where $\mathcal{F}(T) = \mathcal{F}$. Suppose that the price process $\{S(t) : 0 \leq t \leq T\}$ of a certain stock is given by

$$S(t) = \exp \left\{ \int_0^t (1 + u) dW(u) + t - \frac{t^3}{6} \right\}.$$

(a) Show that $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process. (1 pt)

(b) Let $r$ be a constant interest rate. Find a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$. (1.5 pts)

**Proof (a):** First note that $S(t) = \exp \left\{ \int_0^t (1 + u) dW(u) + \int_0^t (1 - \frac{u^2}{2}) du \right\}$. Let

$$X(t) = \int_0^t (1 + u) dW(u) + \int_0^t (1 - \frac{u^2}{2}) du,$$

then $X(0) = 0$ and $dX(t) = (1 + t) dW(t) + (1 - \frac{t^2}{2}) dt$. Consider the continuously differentiable function $f(x) = e^x$. We have $f_x(x) = f_{xx}(x) = e^x$ and $S(t) = f(X(t)) = f_x(X(t)) = f_{xx}(X(t))$. By Itô-Doeblin,

$$d(S(t)) = d(f(X(t))) = S(t) dX(t) + \frac{1}{2} S(t) dX(t) dX(t) = S(t) \left((1 + t) dW(t) + (1 - \frac{1}{2} t^2) dt \right) + \frac{1}{2} S(t)(1 + t)^2 dt.$$ 

Hence, $S(t) = 1 + \int_0^t (\frac{3}{2} + u) S(u) du + \int_0^t (1 + u) S(u) dW(u)$, and therefore $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process.

**Proof (b):** We consider the adapted process $\{\theta(t) : 0 \leq t \leq T\}$ given by $\theta(t) = \frac{3}{2} + \frac{t - r}{1 + t}$, and the random variable

$$Z = \exp \left\{ - \int_0^T \theta(u) dW(u) - \frac{1}{2} \int_0^T \theta^2(u) du \right\}.$$ 

Notice that $\theta(t)$ is bounded on the interval $[0, T]$, hence $\mathbb{E} \left[ \int_0^T \theta^2(u) Z^2(u) du \right] < \infty$. Consider the probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ defined by $\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}$, and the process $\tilde{W}(t) : 0 \leq t \leq T$ is a Brownian motion under $\tilde{\mathbb{P}}$. Therefore $\tilde{W}(t)$ is a martingale under $\tilde{\mathbb{P}}$. Moreover, $\tilde{W}(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$. Therefore $\tilde{W}(t)$ is a martingale under $\tilde{\mathbb{P}}$. Moreover, $\tilde{W}(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$. Therefore $\tilde{W}(t)$ is a martingale under $\tilde{\mathbb{P}}$.
Let $\tilde{W}(t) = \int_0^t \theta(u) \, du + W(t)$. By Girsanov’s Theorem, the process $\{\tilde{W}(t): 0 \leq t \leq T\}$ is a Brownian motion under $\tilde{F}$, and by Itô product rule we have,

$$d(e^{-rt}S(t)) = e^{-rt} dS(t) - re^{-rt} S(t) \, dt$$

$$= e^{-rt}[(\frac{3}{2} + t)S(t) \, dt + (1 + t)S(t) \, dW(t)] - re^{-rt} S(t) \, dt$$

$$= e^{-rt}(\frac{3}{2} + t - r)S(t) \, dt + e^{-rt}(1 + t)S(t) \, dW(t)$$

$$= e^{-rt}(1 + t)S(t) \left(\theta(t) \, dt + dW(t)\right)$$

$$= e^{-rt}(1 + t)S(t)\tilde{dW}(t).$$

This shows that the process $\{e^{-rt}S(t): 0 \leq t \leq T\}$ is an Itô process under $\tilde{P}$ and hence a martingale under $\tilde{P}$.

(4) Let $T$ be a finite time (expiration date) and let $\{(W_1(t), W_2(t): 0 \leq t \leq T)\}$ be a two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\{F(t): 0 \leq t \leq T\}$, where $\mathcal{F} = \mathcal{F}(T)$. Consider two price processes

$$dS_1(t) = S_1(t) \, dt + 0.3S_1(t) \, dW_1(t) + 0.3S_1(t) \, dW_2(t)$$

$$dS_2(t) = 2S_2(t) \, dt + 0.1S_2(t) \, dW_1(t).$$

We assume $S_1(0), S_2(0) > 0$.

(a) Assume that the interest rate is a constant, i.e. $R(t) = r$ for $t > 0$. Find the unique risk-neutral probability $\tilde{P}$, i.e. the probability measure $\tilde{P}$ equivalent to $P$ under which the discounted price processes $\{e^{-rt}S_i(t): 0 \leq t \leq T\}$ are martingales, $i = 1, 2$. (1.5 pts)

(b) Consider a financial derivative with payoff at time $T$ given by $V(T) = \frac{1}{T} \int_0^T S_2(t) \, dt$. Show that the fair price at time 0 of this derivative is given by $V(0) = \frac{S_2(0)}{r^T} (1 - e^{-rT})$. (1.5 pts)

**Proof (a):** We will show that the market price equations have a unique solution. Using the notation of the book we have, $\alpha_1 = 1, \sigma_{11} = 0.3, \sigma_{12} = 0.3, \alpha_2 = 2, \sigma_{21} = 0.1, \sigma_{22} = 0$ and the market price equations are given by

$$1 - r = 0.3\theta_1(t) + 0.3\theta_2(t)$$

$$2 - r = 0.1\theta_1(t).$$

This system has a unique solution given by $\theta_1(t) = 10(2 - r)$ and $\theta_2(t) = \frac{10(2r - 5)}{3}$. Setting

$$Z = \exp\left\{ - \int_0^T \left(\theta_1(t) \, dW_1(t) + \theta_2(t) \, dW_2(t)\right) - \frac{1}{2} \int_0^T \left(\theta_1^2(t) + \theta_2^2(t)\right) \, dt\right\}$$

$$= \exp\left\{ - 10(2 - r)W_1(T) - \frac{10(2r - 5)}{3} W_2(T) - \left(50(2 - r)^2 + \frac{50(2r - 5)^2}{9}\right)T\right\},$$

the desired risk-neutral probability measure is given by $\tilde{P}(A) = \int_A Z \, dP$. To check this, we set

$$\tilde{W}_1(t) = \int_0^t \theta_1(u) \, du + W_1(t) = 10(2 - r)t + W_1(t),$$

and

$$\tilde{W}_2(t) = \int_0^t \theta_2(u) \, du + W_2(t) = \frac{10(2r - 5)}{3} t + W_2(t).$$

By the 2-dimensional Girsanov Theorem the process $\{\tilde{W}_1(t), \tilde{W}_2(t): 0 \leq t \leq T\}$ is a 2-dimensional Brownian motion. Applying Itô product rule on $e^{-rt}S_1(t), e^{-rt}S_2(t)$ and rewriting in terms of $\tilde{W}_1(t), \tilde{W}_2(t)$, we get

$$d(e^{-rt}S_1(t)) = e^{-rt}S_1(t)(0.3d\tilde{W}_1(t) + 0.3d\tilde{W}_2(t))$$

$$d(e^{-rt}S_2(t)) = e^{-rt}S_2(t)0.1d\tilde{W}_1(t),$$
which shows that the discounted price processes are martingales under $\tilde{P}$.

**Proof (b):** From part (a), we see that the Market price equations have a unique solution and hence the risk neutral measure $\tilde{P}$ is unique. By the Second Fundamental Theorem of Asset Pricing, the market is complete and every derivative security can be hedged. Hence, there exists a (self-financing) portfolio $\{X(t) : 0 \leq t \leq T\}$ such that $X(t) = V(t)$ for all $0 \leq t \leq T$. Since $\{e^{-rt}X(t) : 0 \leq t \leq T\}$, $\{e^{-rt}S_1(t) : 0 \leq t \leq T\}$ and $\{e^{-rt}S_2(t) : 0 \leq t \leq T\}$ are all martingales under $\tilde{P}$, we see that the price at time zero is given by

$$X(0) = V(0) = \tilde{E}[e^{-rT}V(T)] = \tilde{E}\left[\frac{e^{-rT}}{T} \int_0^T S_2(t) \, dt\right] = \frac{e^{-rT}}{T} \int_0^T e^{rt} \tilde{E}[e^{-rt}S_2(t)] \, dt = \frac{e^{-rT}}{T} \int_0^T e^{rt} S_2(0) \, dt = \frac{e^{-rT}S_2(0)}{rT} (e^T - 1) = \frac{S_2(0)}{rT} (1 - e^{-rT}),$$

the third equality follows from Fubini’s (in fact Tonelli’s) Theorem since $S_2(t) = S_2(0) \exp \left\{0.1W_1(t) + (2 - \frac{1}{2}(0.1)^2)t\right\} > 0$ and the fourth equality follows from the fact that $\{e^{-rt}S_2(t) : 0 \leq t \leq T\}$ is a martingales under $\tilde{P}$. 