Solutions Mid-Term: Inleiding Financiële Wiskunde 2019-2020

(1) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n \in \mathbb{N}}$ be a sequence of pairwise independent sets in $\mathcal{F}$ (i.e. $P(A_n \cap A_m) = P(A_n)P(A_m)$ for $n \neq m$) satisfying $P(A_n) = 1/2$ for all $n \geq 1$. Let $I_n$ be the indicator function of the set $A_n$ and $\sigma(I_n)\sigma$ the $\sigma$-algebra generated by the random variable $I_n$, $n \geq 1$.

(a) Prove that $\sigma(I_n) = \{\emptyset, \Omega, A_n, A_n^c\}$ and that the $\sigma$-algebras $\sigma(I_n)$ and $\sigma(I_m)$ are independent whenever $n \neq m$, i.e. $P(C \cap D) = P(C)P(D)$ for any $C \in \sigma(I_n)$ and any $D \in \sigma(I_m)$. Conclude that $I_1, I_2, \cdots$ is a pairwise independent sequence. (1.5 pts)

(b) For $n \geq 1$, define $X_n = 2I_n - 1$. Set $M_0 = 0$, $M_n = \sum_{k=1}^{n} 2^{k-1}X_k$ for $n \geq 1$ and let $Y_n = M_n^2 - \frac{4^n - 1}{3}$ for $n \geq 0$. Consider the filtration $\{\mathcal{F}(n) : n \geq 0\}$ where $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(n) = \sigma(I_1, \cdots, I_n)$ is the smallest $\sigma$-algebra containing all sets of the form $\{I_j \in B\}$ for any Borel set $B$ and any $1 \leq j \leq n$. Prove that the process $\{Y_n : n \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(n) : n \geq 0\}$. (1.5 pts)

Proof (a): By definition $\sigma(I_n) = \{\{I_n \in B\} : B$ is a Borel set$\}$. Since $I_n$ takes only the values 0 and 1, we see that

$$\{I_n \in B\} := \begin{cases} \emptyset & \text{if } 0, 1 \notin B \\ A_n^c & \text{if } 0 \in B, \text{ and } 1 \notin B \\ A_n & \text{if } 1 \in B, \text{ and } 0 \notin B \\ \Omega & \text{if } 0, 1 \in B. \end{cases}$$

Thus, $\sigma(I_n) = \{\emptyset, \Omega, A_n, A_n^c\}$.

Next we need to show that the $\sigma$-algebras $\sigma(I_n)$ and $\sigma(I_m)$ are independent whenever $n \neq m$, i.e. $P(C \cap D) = P(C)P(D)$ for any $C \in \sigma(I_n)$ and any $D \in \sigma(I_m)$. First note that $\sigma(I_n) = \{\emptyset, \Omega, A_n, A_n^c\}$ and $\sigma(I_m) = \{\emptyset, \Omega, A_m, A_m^c\}$. If $C$ or $D$ is either $\emptyset$ or $\Omega$, then the result is trivially true. So we only need to consider the case $C \in \{A_n, A_n^c\}$ and $D \in \{A_m, A_m^c\}$. By hypothesis, $P(A_n \cap A_m) = P(A_n)P(A_m)$. For the other cases, we first note that

$$P(A_n) = P(A_n \cap A_m^c) + P(A_n \cap A_m) = P(A_n \cap A_m^c) + P(A_n)P(A_m),$$

implying

$$P(A_n \cap A_m^c) = P(A_n) - P(A_n)P(A_m) = P(A_n) \left(1 - P(A_m)\right) = P(A_n)P(A_m^c).$$

Similarly,

$$P(A_m) = P(A_m \cap A_n^c) + P(A_m \cap A_n) = P(A_m \cap A_n^c) + P(A_m)P(A_n),$$

leading to $P(A_m \cap A_n^c) = P(A_m)P(A_n^c)$. Finally,

$$P\left(A_n^c \cap A_m^c\right) = P\left((A_n \cup A_m)^c\right) = 1 - P\left(A_n \cup A_m\right)$$

$$= 1 - (P(A_n) + P(A_m) - P(A_n \cap A_m))$$

$$= 1 - (P(A_n) + P(A_m) - P(A_n)P(A_m))$$

$$= (1 - P(A_n))(1 - P(A_m))$$

$$= P(A_n^c)P(A_m^c).$$

This shows that $\sigma(I_n)$ and $\sigma(I_m)$ are independent whenever $n \neq m$. Since by definition two random variables $X$ and $Y$ are independent if $\sigma(X)$ and $\sigma(Y)$ are independent, we conclude that the sequence $I_1, I_2, \cdots$ is pairwise independent.
Proof (b): First note that
\[ X_n(\omega) = \begin{cases} 
1 & \omega \in A_n \\
-1 & \omega \notin A_n.
\end{cases} \]
From here we see that \( F(n) = \sigma(X_1, \ldots, X_n) \) and \( E(X_n) = 2P(A_n) - 1 = 0 \) for all \( n \geq 1 \). Since \( F(1) \subset F(2) \subset \cdots \subset F(n) \), we see that \( M_n \) is \( F(n) \)-measurable implying that \( Y_n \) is \( F(n) \)-measurable and hence the process \( \{Y_n : n \geq 0\} \) is adapted to the filtration \( \{F(n) : n \geq 0\} \). To show that the process \( \{Y_n : n \geq 0\} \) is a martingale, it is enough to show that \( E[Y_n+1|F(n)] = Y_n \), for then by the repeated application of the iterated conditioning property we will have \( E[Y_n|F(m)] = Y_m \) for any \( m < n \) (see the solutions of the Mock Mid-term). Note that
\[ M_n^2+1 = (M_n + 2^nX_n+1)^2 = M_n^2 + 2^nM_nX_n+1 + 4^n X_n^2+1. \]
Since \( X_n+1 \) is independent of \( F(n) \), we have \( E[X_n+1|F(n)] = E[X_n+1|E] = 0 \) and \( E[X_n^2|F(n)] = E[X_n^2+1|E] = 1 \). By linearity of the conditional expectation, the \( F(n) \)-measurability of \( M_n \) and the take out what you know property, we have
\[ E[M_n^2|F(n)] = E[M_n^2 + 2^nM_nE[X_n+1] + 4^nE[X_n^2+1] = M_n^2 + 4^n. \]
Thus,
\[ E[Y_n+1|F(n)] = E[M_n^2+1|F(n)] - \frac{(4^n+1-1)}{3} = M_n^2 + 4^n - \frac{(4^n+1-1)}{3} = M_n^2 - \frac{(4^n-1)}{3} = Y_n. \]
Therefore, \( \{Y_n : n \geq 0\} \) is a martingale with respect to the filtration \( \{F(n) : n \geq 0\} \).

(2) Let \( \{W(t) : t \geq 0\} \) be a Brownian motion defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and let \( \{\mathcal{F}(t) : t \geq 0\} \) be a filtration for the Brownian motion. Define a process \( \{X(t) : t \geq 0\} \) by \( X(t) = e^{W(t)-\frac{1}{2}t^3+1}, t \geq 0 \).

(a) Prove that \( \mathbb{P}(X(1) > 1) = 1/2 \). (1 pt)
(b) Derive an expression for \( \text{Var}[X(t)] \), the variance of \( X(t) \). (1.5 pts)
(c) For \( s < t \), determine an expression for \( E[X(t)|\mathcal{F}(s)] \). (1.5 pts)

Proof (a): We have \( X(1) = e^{W(1)} \), with \( W(1) \) a standard normal random variable (so mean zero and variance 1). Thus,
\[ \mathbb{P}(X(1) > 1) = \mathbb{P}(\ln(X(1) > 0) = \mathbb{P}(W(1) > 0) = 1 - \mathbb{P}(W(1) \leq 0) = 1 - N(0) = 1/2, \]
where \( N \) denotes the cumulative distribution function of the standard normal distribution.

Proof (b): We first calculate the expectation of \( X(t) \), we have
\[ E[X(t)] = e^{-\frac{1}{2}t^3+1}E[e^{W(t)}] = e^{-\frac{1}{2}t^3+1}e^{\frac{1}{2}t^3} = e^{-\frac{1}{2}t^3+1}, \]
where in the second equality we used that the moment generating function of the \( N(0, t) \) random variable \( W(t) \) has value \( E[e^{uW(t)}] = e^{\frac{1}{2}u^2t} \) (in our case \( u = t \)). Next we calculate the expectation of \( X^2(t) = e^{2tW(t)-2t^3+2} \),
\[ E[X^2(t)] = e^{-2t^3+2}E[e^{2tW(t)}] = e^{-2t^3+2}e^{\frac{1}{2}4t^3} = e^2. \]
Thus,
\[ \text{Var}(X(t)) = E[X^2(t)] - (E[X(t)])^2 = e^2 - e^{-\frac{1}{2}t^3+1} = 2(1 - e^{-\frac{1}{2}t^3}). \]
Proof (c): Using the fact that \( W(s) \) is \( \mathcal{F}(s) \)-measurable and that \( W(t) - W(s) \) is independent of \( \mathcal{F}(s) \), we have by the properties of conditional expectation,

\[
E[X(t) | \mathcal{F}(s)] = e^{-t^3 + 1} E[e^{tW(t)} | \mathcal{F}(s)] \\
= e^{-t^3 + 1} E[e^{t(W(t) - W(s)) + tW(s)} | \mathcal{F}(s)] \\
= e^{-t^3 + 1} e^{tW(s)} E[e^{t(W(t) - W(s))} | \mathcal{F}(s)] \\
= e^{-t^3 + 1} e^{tW(s)} E[e^{t(W(t) - W(s))}] \\
= e^{-t^3 + 1} e^{tW(s)} e^{\frac{1}{2}t^2(t-s)} \\
= e^{tW(s) - \frac{1}{2}t^2(t+s)+1},
\]

where in the first equality we used the linearity of the conditional expectation, in the third equality we used the property take out what you know, in the fourth equality we used the independence of \( W(t) - W(s) \) and \( \mathcal{F}(s) \) and in the fifth equality we used the fact that the moment generating function of \( W(t) - W(s) \) is given by \( E[e^{u(W(t) - W(s))}] = e^{\frac{1}{2}u^2(t-s)} \) for \( u \in \mathbb{R} \).

(3) Let \( \{W(t) : t \geq 0\} \) and \( \{V(t) : t \geq 0\} \) be two independent Brownian motions defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). By independence we mean that \( W(t) \) and \( V(s) \) are independent for all \( s, t > 0 \). Let \( 0 < \rho < 1 \) be a positive real number and define a process \( \{Z(t) : t \geq 0\} \) by \( Z(t) = \rho W(t) + \sqrt{1-\rho^2} V(t) \). Prove that the process \( \{Z(t) : t \geq 0\} \) is a Brownian motion. (3 pts)

(Hint: if \( X \) and \( Y \) are independent normally distributed random variables with \( X \) being \( \mathcal{N}(\mu_1, \sigma_1^2) \) and \( Y \) being \( \mathcal{N}(\mu_2, \sigma_2^2) \), then \( X + Y \) is normally \( \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \) distributed).

Proof: We check that the process \( \{Z(t) : t \geq 0\} \) satisfies all the properties of a Brownian motion. We have

(i) \( Z(0) = \rho W(0) + \sqrt{1-\rho^2} V(0) = 0. \)

(ii) Since both \( \{W(t) : t \geq 0\} \) and \( \{V(t) : t \geq 0\} \) have continuous paths and a linear combination of continuous functions is continuous, we that the process has continuous paths.

(iii) Let \( 0 = t_0 < t_1 < \cdots < t_m \), then \( W(t_{i+1}) - W(t_i) \) is independent of \( W(t_{j+1}) - W(t_j) \) and \( V(t_{i+1}) - V(t_i) \) is independent of \( V(t_{j+1}) - V(t_j) \) for all \( i \neq j \). Furthermore, \( W(t_{i+1}) - W(t_i) \) is independent of \( V(t_{j+1}) - V(t_j) \) for all \( i, j = 1, \cdots , m \). Thus the increments \( Z(t_1) - Z(t_0), \cdots , Z(t_m) - Z(t_{m-1}) \) are independent.

(iv) Let \( s < t \), then \( Z(t) - Z(s) = \rho(W(t) - W(s)) + \sqrt{1-\rho^2}(V(t) - V(s)) \). By hypothesis the random variables \( W(t) - W(s) \) and \( V(t) - V(s) \) are independent and both are normally \( \mathcal{N}(0, \sigma^2) \) distributed. Thus, \( \rho(W(t) - W(s)) \) and \( \sqrt{1-\rho^2}(V(t) - V(s)) \) are independent with \( \rho(W(t) - W(s)) \) normally \( \mathcal{N}(0, \rho^2(t-s)) \) distributed and \( \sqrt{1-\rho^2}(V(t) - V(s)) \) normally \( \mathcal{N}(0, (1-\rho^2)(t-s)) \) distributed. Using the hint we have that \( Z(t) - Z(s) \) is normally \( \mathcal{N}(0, t-s) \).

Therefore, \( \{Z(t) : t \geq 0\} \) is a Brownian motion.