## Measure and Integration: Solutions Retake Exam 2020-21

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra and $\lambda$ is Lebesgue measure. For $n \geq 1$, let $u_n(x) = \mathbb{I}_{[0,1-2^{-n})}(x)\cos(e^{-x/n})\,x^2$.

(a) Prove that $\lim_{n\to\infty} \int u_n\, d\lambda = \frac{1}{3}\cos(1)$. (1 pt)

(b) Let $1 < p < \infty$, prove that $\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| < \infty$ $\lambda$ a.e. (1.5 pts)

**Proof (a)** Let $u(x) = \mathbb{I}_{[0,1)}x^2\cos(1)$, since $u$ is Riemann integrable, then $u \in \mathcal{L}^1(\lambda)$ and

$$\int u\, d\lambda = (R)\int_0^1 x^2\cos(1)\, dx = \frac{1}{3}\cos(1).$$

We have $\lim_{n\to\infty} u_n(x) = \mathbb{I}_{[0,1)}(x)x^2\cos(1) = \mathbb{I}_{[0,1)}(x)x^2 = u(x)$ and $|u_n(x)| \leq u(x)$. Thus, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int u_n\, d\lambda = \int \lim_{n\to\infty} u_n\, d\lambda = \int u\, d\lambda = \frac{1}{3}\cos(1).$$

**Proof (b)** For $1 < p < \infty$, we have

$$\begin{aligned}\int |u_n|^p\, d\lambda &= \int \mathbb{I}_{[0,1-2^{-n})}(x)|\cos(e^{-x/n})|^p\, x^{2p}\, d\lambda(x)\\ &\leq \int \mathbb{I}_{[0,1)}(x)\, x^{2p}\, d\lambda(x)\\ &= (R)\int_0^1 x^{2p}\, dx\\ &= \frac{1}{2p+1}.\end{aligned}$$

Thus, by Corollary 9.9 and the fact that $1 < p < \infty$ we have

$$\begin{aligned}\int \sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p d\lambda &= \sum_{n=1}^{\infty}\int \frac{|u_n|^p}{n^p}d\lambda\\ &\leq \frac{1}{2p+1}\sum_{n=1}^{\infty}\frac{1}{n^p}\\ &< \infty.\end{aligned}$$

By Corollary 11.6, $\sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p < \infty$ $\lambda$ a.e. Since

$$\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| \leq \sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p,$$

it follows that $\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| < \infty$ $\lambda$ a.e.

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $1 < p, q < \infty$ conjugate numbers, i.e. $1/p + 1/q = 1$ and $u \in \mathcal{L}^p(\mu)$ with $||u||_p > 0$.

(a) Define

$$v(x) = \left(\frac{u(x)}{||u||_p}\right)^{p-1}$$

Prove that $v \in \mathcal{L}^q(\mu)$ and $||v||_q = 1$. (1 pt)

(b) Prove that $\int |uv|\, d\mu = ||u||_p$. (1 pt)

**Proof (a)**: Note that $q(p-1) = p$, hence $|v(x)|^q = \dfrac{|u(x)|^p}{\|u\|_p^p}$. We have,

$$\int |v|^q \, d\mu = \int \frac{|u|^p}{\|u\|_p^p} \, d\mu = 1.$$

So $\|v\|_q = 1$ and $v \in \mathcal{L}^q(\mu)$.

**Proof (b)**: Note that

$$|uv(x)| = \frac{|u(x)|^p}{\|u\|_p^{p-1}}$$

Thus,

$$\int |uv| \, d\mu = \int \frac{|u|^p}{\|u\|_p^{p-1}} \, d\mu = \|u\|_p.$$

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel $\sigma$-algebra and $\lambda$ is Lebesgue measure. Let $u \in \mathcal{L}^1(\lambda)$ and define for $h > 0$, the function $u_h(x) = \dfrac{1}{h} \displaystyle\int_{[x,x+h]} u(t) \, d\lambda(t)$.
 (a) Show that $u_h$ is Borel measurable for all $h > 0$. (1 pt)

 (b) Show that $u_h \in \mathcal{L}^1(\lambda)$ and $\|u_h\|_1 = \|u\|_1$. (1.5 pt)

**Proof (a)**: For $h > 0$, define $v_h(t,x) = \frac{1}{h}\mathbb{I}_{[x,x+h]}(t)u(t)$, then $v_h$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ measurable. Applying Tonelli's Theorem 14.8 (edition 1, Theorem 13.8(ii) ) to the positive and negative parts of the function $v_h$, we have that the functions

$$x \to \int v^+(t,x) \, d\lambda(t) = u_h^+(x), \text{ and } x \to \int v^-(t,x) \, d\lambda(t) = u_h^-(x)$$

are $\mathcal{B}(\mathbb{R})$ measurable. Hence, $u_h$ is Borel measurable for all $h > 0$.

**Proof (b)**: By Tonelli's Theorem, we have

$$\int \int \frac{1}{h}\mathbb{I}_{[x,x+h]}(t)|u(t)| \, d\lambda(t) \, d\lambda(x) = \int \int \frac{1}{h}\mathbb{I}_{[t-h,t]}(x)|u(t)| \, d\lambda(x) \, d\lambda(t) = \int |u(t)| \, d\lambda(t) < \infty.$$

Thus,

$$\|u_h\|_1 = \int |u_h(x)| \, d\lambda(x) = \int \int \frac{1}{h}\mathbb{I}_{[x,x+h]}(t)|u(t)| \, d\lambda(x) \, d\lambda(t) = \int |u(x)| \, d\lambda(x) = \|u\|_1.$$

This shows that $u_h \in \mathcal{L}^1(\lambda)$ and $\|u_h\|_1 = \|u\|_1$.

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(u_n)_n \subset \mathcal{M}^+(\mathcal{A})$ a sequence of non-negative real-valued measurable functions such that $\lim_{n\to\infty} u_n = u$ , where $u$ is non-negative measurable function. Assume that

$$\lim_{n\to\infty} \int_X u_n \, d\mu = \int_X u \, d\mu < \infty,$$

and let $A \in \mathcal{A}$.
 (a) Prove that

$$\int_A u \, d\mu \geq \limsup_{n\to\infty} \int_A u_n \, d\mu.$$

 (Hint: apply Fatou's lemma to the sequence $v_n = u_n - \mathbb{I}_A u_n$.) (2 pts)
 (b) Prove that

$$\int_A u \, d\mu = \lim_{n\to\infty} \int_A u_n \, d\mu.$$

 (1 pt)

**Proof (a)** Let $v_n = u_n - \mathbb{I}_A u_n$, then by hypothesis

$$u - \mathbb{I}_A u = \lim_{n\to\infty} v_n = \liminf_{n\to\infty} v_n.$$

By Fatou's Lemma, and the linearity of the integrals we have

$$\int u \, d\mu - \int \mathbb{I}_A u \, d\mu \leq \liminf_{n\to\infty} \int v_n \, d\mu \leq \liminf_{n\to\infty} \Big( \int u_n \, d\mu - \int \mathbb{I}_A u_n \, d\mu \Big).$$

Since

$$\int u\,d\mu = \lim_{n\to\infty} \int u_n\,d\mu = \liminf_{n\to\infty} \int u_n\,d\mu,$$

one has that

$$\begin{aligned}\liminf_{n\to\infty}\Big(\int u_n\,d\mu - \int \mathbb{I}_A u_n\,d\mu\Big) &= \lim_{n\to\infty}\int u_n\,d\mu + \liminf_{n\to\infty}(-\int \mathbb{I}_A u_n\,d\mu)\\ &= \int u\,d\mu - \limsup_{n\to\infty}\int \mathbb{I}_A u_n, d\mu.\end{aligned}$$

The above gives

$$\int u\,d\mu - \int \mathbb{I}_A u\,d\mu \le \int u\,d\mu - \limsup_{n\to\infty}\int \mathbb{I}_A u_n, d\mu.$$

Subtracting $\int u\,d\mu < \infty$ from both sides leads to

$$\int \mathbb{I}_A u\,d\mu \ge \limsup_{n\to\infty}\int \mathbb{I}_A u_n\,d\mu.$$

Equivalently,

$$\int_A u\,d\mu \ge \limsup_{n\to\infty}\int_A u_n\,d\mu.$$

**Proof (b)** By Fatou's Lemma, we have

$$\int \mathbb{I}_A u\,d\mu = \int \lim_{n\to\infty} \mathbb{I}_A u_n\,d\mu \le \liminf_{n\to\infty}\int \mathbb{I}_A u_n\,d\mu.$$

Combining with part (a), we have

$$\int \mathbb{I}_A u\,d\mu = \lim_{n\to\infty}\int \mathbb{I}_A u_n\,d\mu.$$