Measure and Integration: Retake Exam 2020-21

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure. For $n \geq 1$, let $u_n(x) = \mathbb{I}_{[0,1-2^{-n})}(x) \cos(e^{-x/n})\, x^2$.

(a) Prove that $\lim_{n\to\infty} \int u_n \, d\lambda = \frac{1}{3}\cos(1)$. (1 pt)

(b) Let $1 < p < \infty$, prove that $\left| \sum_{n=1}^{\infty} \left(\frac{u_n}{n}\right)^p \right| < \infty$ λ a.e. (1.5 pts)

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $1 < p, q < \infty$ conjugate numbers, i.e. $1/p + 1/q = 1$ and $u \in \mathcal{L}^p(\mu)$ with $\|u\|_p > 0$.

(a) Define

$$v(x) = \left(\frac{u(x)}{\|u\|_p}\right)^{p-1}$$

Prove that $v \in \mathcal{L}^q(\mu)$ and $\|v\|_q = 1$. (1 pt)

(b) Prove that $\int |uv| \, d\mu = \|u\|_p$. (1 pt)

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $u \in \mathcal{L}^1(\lambda)$ and define for $h > 0$, the function $u_h(x) = \frac{1}{h} \int_{[x,x+h]} u(t)\, d\lambda(t)$.

(a) Show that u_h is Borel measurable for all $h > 0$. (1 pt)

(b) Show that $u_h \in \mathcal{L}^1(\lambda)$ and $\|u_h\|_1 = \|u\|_1$. (1.5 pt)

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(u_n)_n \subset \mathcal{M}^+(\mathcal{A})$ a sequence of non-negative real-valued measurable functions such that $\lim_{n\to\infty} u_n = u$, where u is non-negative measurable function. Assume that

$$\lim_{n\to\infty} \int_X u_n \, d\mu = \int_X u \, d\mu < \infty,$$

and let $A \in \mathcal{A}$.

(a) Prove that

$$\int_A u \, d\mu \geq \limsup_{n\to\infty} \int_A u_n \, d\mu.$$

(Hint: apply Fatou's lemma to the sequence $v_n = u_n - \mathbb{I}_A u_n$.) (2 pts)

(b) Prove that

$$\int_A u \, d\mu = \lim_{n\to\infty} \int_A u_n \, d\mu.$$

(1 pt)