(1) Let $X$ be a set and $\mu, \nu$ two outer measures on $X$, i.e. $\mu, \nu : \mathcal{P}(X) \to [0, \infty]$ satisfying the three properties:

(i) $\mu(\emptyset) = \nu(\emptyset) = 0$,

(ii) if $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and $\nu(A) \leq \nu(B)$ ($\mu$ and $\nu$ are monotone),

(iii) if $(A_n)$ is a sequence in $\mathcal{P}(X)$, then $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$ and $\nu(\bigcup_n A_n) \leq \sum_n \nu(A_n)$.

Define $\rho : \mathcal{P}(X) \to [0, \infty]$ by $\rho(A) = \max(\mu(A), \nu(A))$. Show that $\rho$ is an outer measure on $X$, i.e. satisfies properties (i), (ii) and (iii). (2 pts)

**Proof:** $\rho(\emptyset) = 0$ is immediate since $\max(0,0) = 0$. The second property is also immediate since if $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and $\nu(A) \leq \nu(B)$, hence $\rho(A) = \max(\mu(A), \nu(A)) \leq \max(\mu(B), \nu(B)) = \rho(B)$.

Now let $(A_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}(X)$, then

$$\rho\left(\bigcup_{n=1}^{\infty} A_n\right) = \max\left(\mu\left(\bigcup_{n=1}^{\infty} A_n\right), \nu\left(\bigcup_{n=1}^{\infty} A_n\right)\right)$$

$$\leq \max\left(\sum_{n=1}^{\infty} \mu(A_n), \sum_{n=1}^{\infty} \nu(A_n)\right)$$

$$\leq \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n))$$

$$= \sum_{n=1}^{\infty} \rho(A_n),$$

where the second inequality follows from the fact that $\sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n))$, and $\sum_{n=1}^{\infty} \nu(A_n) \leq \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n))$. Thus, $\rho$ is an outer measure on $X$.

(2) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the Borel $\sigma$-algebra restricted to $[0,1]$ and $\lambda$ is the restriction of Lebesgue measure on $[0,1]$. Define a map $u : [0,1] \to [0,1]$ by $u(x) = 2x \cdot 1_{[0,1/2]} + (2-2x) \cdot 1_{[1/2,1]}$, where $1_A$ denotes the indicator function of the set $A$.

(a) Show that $u$ is $\mathcal{B}([0,1])/\mathcal{B}([0,1])$ measurable, and determine the image measure $\mu(u^{-1}(A)) = \lambda \circ u^{-1}$. (2 pts)

(b) Let $C = \left\{ A \in \mathcal{B}([0,1]) : \lambda(u^{-1}(A) \Delta A) = 0 \right\}$. Show that $C$ is a $\sigma$-algebra. (Note that $u^{-1}(A) \Delta A = \left( u^{-1}(A) \setminus A \right) \cup \left( A \setminus u^{-1}(A) \right)$.) (2.5 pts)

**Proof:** To show $u$ is $\mathcal{B}([0,1])/\mathcal{B}([0,1])$ measurable, it is enough to consider inverse images of intervals of the form $[a,b) \subset [0,1]$. Now,

$$u^{-1}([a,b)) = \left[ \frac{a}{2}, \frac{b}{2} \right) \cup \left[ \frac{2-b}{2}, \frac{2-a}{2} \right) \in \mathcal{B}([0,1]).$$

Thus, $u$ is measurable. Another quick way of showing measurability is to notice that the functions $x \to 2x$ and $x \to 2-2x$ are continuous and hence Borel measurable. Furthermore $1_{[0,1/2)}$ and $1_{[1/2,1]}$...
are Borel measurable since \([0, \frac{1}{2}), [\frac{1}{2}, 1] \in \mathcal{B}([0, 1])\). Since products and sums of measurable functions are measurable, we see that \(u\) is also Borel measurable.

We claim that \(u(\lambda) = \lambda\). To prove this, we use Theorem 5.7. Notice that \(\mathcal{B}([0, 1])\) is generated by the collection \(\mathcal{G} = \{[a, b] : 0 \leq a \leq b \leq 1\} \cup \{\emptyset\}\) which is closed under finite intersections. Now,

\[
\begin{align*}
\lambda((a, b)) &= \lambda(u^{-1}([a, b])) \\
&= \lambda(\bigcup_{n} \left(\left[\frac{a}{2^n}, \frac{b}{2^n}\right]\right)) + \lambda(\bigcup_{n} \left(\left[\frac{2 - b}{2^n}, \frac{2 - a}{2^n}\right]\right)) \\
&= b - a - \lambda([a, b]).
\end{align*}
\]

Since the constant sequence \((0, 1]\)) is exhausting, belongs to \(\mathcal{G}\) and \(\lambda([0, 1]) = u(\lambda([0, 1]) = 1 < \infty\), we have by Theorem 5.7 that \(u(\lambda) = \lambda\).

**Proof(b):** We check the three conditions for a collection of sets to be a \(\sigma\)-algebra. Firstly, the empty set \(\emptyset \in \mathcal{B}([0, 1])\) and \(u^{-1}(\emptyset) = \emptyset\), hence \(\lambda(u^{-1}(\emptyset)) = \lambda(\emptyset) = 0\), so \(\emptyset \in \mathcal{C}\). Secondly, Let \(A \in \mathcal{C}\), then \(\lambda(u^{-1}(A) \Delta A) = 0\). We have

\[
\lambda\left(u^{-1}(A)^c \Delta A^c\right) = \lambda\left(u^{-1}(A) \Delta A\right) = 0,
\]

with \(A^c \in \mathcal{B}([0, 1])\), hence \(A^c \in \mathcal{C}\). Thirdly, let \((A_n)\) be a sequence in \(\mathcal{C}\), then \(A_n \in \mathcal{B}([0, 1])\) and \(\lambda\left(u^{-1}(A_n) \Delta A_n\right) = 0\) for each \(n\). Since \(\mathcal{B}([0, 1])\) is a \(\sigma\)-algebra, we have \(\bigcup_n A_n \in \mathcal{B}([0, 1])\), and

\[
u^{-1}\left(\bigcup_n A_n\right) = \bigcup_n u^{-1}(A_n).
\]

An easy calculation shows that

\[
u^{-1}\left(\bigcup_n A_n\right) \Delta \bigcup_n A_n \subseteq \bigcup_n \left(u^{-1}(A_n) \Delta A_n\right).
\]

By \(\sigma\)-subadditivity and monotonicity of \(\lambda\), we have

\[
\lambda\left(u^{-1}\left(\bigcup_n A_n\right) \Delta \bigcup_n A_n\right) \leq \lambda\left(\bigcup_n \left(u^{-1}(A_n) \Delta A_n\right)\right) \leq \sum_n \lambda\left(u^{-1}(A_n) \Delta A_n\right) = 0.
\]

Thus, \(\bigcup_n A_n \in \mathcal{C}\). This shows that \(\mathcal{C}\) is a \(\sigma\)-algebra.

(3) Let \((X, \mathcal{A})\) be a measurable space and \((A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}\), a partition of \(X\), i.e. \(A_n \in \mathcal{A}\) are pairwise disjoint and \(X = \bigcup_{n \in \mathbb{N}} A_n\). Consider the function \(u : X \to \mathbb{R}\) defined by

\[
u(x) = \sum_{j \in \mathbb{N}} 2^j \cdot 1_{A_j}(x).
\]

(a) Show that \(u \in \mathcal{M}(\mathcal{A})\), i.e. \(u\) is \(\mathcal{A}/\mathcal{B}(\mathbb{R})\) measurable. (1.5 pt)

(b) Recall that \(\sigma(u) = \left\{u^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\right\}\) is the smallest \(\sigma\)-algebra on \(X\) making \(u\) Borel measurable. Prove that

\[
s(u) = \sigma(\{A_n : n \in \mathbb{N}\}),
\]

where \(\sigma(\{A_n : n \in \mathbb{N}\})\) is the smallest \(\sigma\)-algebra generated by the countable collection \(\{A_n : n \in \mathbb{N}\}\). (2 pts)

**Proof(a):** For each \(n \in \mathbb{N}\), define \(u_n(x) = \sum_{j=1}^{n} 2^j \cdot 1_{A_j}(x)\). Since \(A_j \in \mathcal{A}\), we see that \((u_n)_{n \in \mathbb{N}}\) is an increasing sequence of non-negative measurable simple functions, i.e. \(u_n \in \mathcal{E}^+(\mathcal{A}) \subseteq \mathcal{M}^+(\mathcal{A})\), and

\[
u(x) = \lim_{n \to \infty} u_n(x) = \sup_{n \in \mathbb{N}} u_n(x).
\]

By Corollary 8.10, it follows that \(u \in \mathcal{M}(\mathcal{A})\), i.e. \(\mathcal{A}/\mathcal{B}(\mathbb{R})\) measurable.

**Proof(b):** Note that \(A_n = u^{-1}(\{2^n\})\) and \(\{2^n\} \in \mathcal{B}(\mathbb{R})\). Hence, \(A_n \in \sigma(u)\) for all \(n \in \mathbb{N}\), implying that \(\sigma(\{A_n : n \in \mathbb{N}\}) \subseteq \sigma(u)\). We now prove the reverse containment. Let \(C \in \sigma(u)\), then
\( C = u^{-1}(B) \) for some Borel set \( B \in \mathcal{B}(\mathbb{R}) \). Set \( N(B) = \{ n \in \mathbb{N} : 2^n \in B \} \) By definition of \( u \), we see that
\[
C = u^{-1}(B) = \bigcup_{n \in N(B)} A_n \in \sigma(\{A_n : n \in \mathbb{N}\}).
\]
This shows that \( \sigma(u) \subseteq \sigma(\{A_n : n \in \mathbb{N}\}) \) and hence \( \sigma(u) = \sigma(\{A_n : n \in \mathbb{N}\}) \).