

WISB263 Mathematical Statistics

Exam V2

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Exercise 1 (25 p.) The average energy expenditures for eight elderly subjects were estimated on the basis of information received from a battery-powered heart rate monitor that each subject wore. Two overall averages were calculated for each subject, one for the summer months and one for the winter months, as shown in the following table. Let μ_D denote the difference between the summer and winter energy expenditure populations.

V2			
Average daily energy expenditure			
Subject	Summer x_i	Winter y_i	
1	1424	1458	
2	1501	1353	
3	1495	2209	
4	1739	1804	
5	2031	1912	
6	934	1366	
7	1401	1598	
8	1339	1406	

- (10 p.) Perform a hypothesis test "TEST 1" for $(H_0) : \mu_D = 0$ against $(H_1) : \mu_D \neq 0$ at $\alpha = 0.05$ and determine the exact rejection region. Is there a smallest level of significance to reject H_0 ?
- (10 p.) Perform a hypothesis test "TEST 2" for $(H_0) : \mu_D = 0$ against $(H_1) : \mu_D \neq 0$ now based on the assumption that the samples are normally distributed. The sample of summer months expenditures X_1, \dots, X_8 has distribution $N(\mu_X, \sigma_X^2)$ and the sample of winter months expenditures Y_1, \dots, Y_8 has distribution $N(\mu_Y, \sigma_Y^2)$ with σ_X, σ_Y unknown and $\mu_D = \mu_X - \mu_Y$. For which α do you reject/accept H_0 ? Compute the p-value. Determine the rejection region for $\alpha = 0.05$.
- (5 p.) Compare the tests "TEST 1" and "TEST 2". Which one is more appropriate?

Exercise 2 (25 p.) Consider an i.i.d. sample X_1, \dots, X_n on some probability space with common density given by

$$h_\theta(x_1) = \frac{g(\theta)}{x_1^4}, \quad x_1 \geq \theta,$$

where $\theta > 0$.

1. (2 p.) Determine $g(\theta)$ as a function of θ .
2. (5 p.) Determine the MLE $\hat{\theta}_n$ of θ and $\hat{\theta}_{n,1}$ for θ^3 .
3. (5 p.) Calculate the cumulative distribution function of $\hat{\theta}_n$ for all $x \in \mathbb{R}$ and determine its density.
4. (4 p.) Is $\hat{\theta}_n$ biased? Show that $\hat{\theta}_n$ is asymptotically unbiased.
5. (4 p.) Find the MoM estimator θ_n^* of θ and compute the variance.
6. (5 p.) Which estimator is more efficient for $n \geq 2$? Can we apply Cramer-Rao Theorem in this case?

Exercise 3 (25 p.) Assume we have an estimator \hat{X}_n for a parameter $\theta > 0$ of the density of an i.i.d. sample X_1, \dots, X_n . The density of the estimator is

$$f_\theta(x) = 3n \frac{\theta^{3n}}{x^{3n+1}}, \quad x > \theta,$$

which means that $\mathbb{P}(\hat{X}_n \in (a, b)) = \int_a^b f_\theta(x) dx$ for $\theta < a < b$. The expected value is equal to $\mathbb{E}(\hat{X}_n) = \frac{3n\theta}{3n-1}$ and variance $Var(\hat{X}_n) = \frac{3n\theta^2}{(3n-2)(3n-1)^2}$.

1. (4 p.) Show that \hat{X}_n is consistent.
2. (4 p.) Prove that $\frac{\hat{X}_n - \mathbb{E}(\hat{X}_n)}{\sqrt{Var(\hat{X}_n)}} \xrightarrow{d} Y$ as $n \rightarrow \infty$, where Y has density $f_Y(y) = e^{-(1+y)}$. (Hint: You can use that $\lim_{n \rightarrow \infty} \left(1 + \frac{1+a}{n}\right)^{-n} = \lim_{n' \rightarrow \infty} \left(1 + \frac{1+a\sqrt{\frac{n'}{n'-2}}}{n'-1}\right)^{-n'}$, where $n' = c \cdot n$ for some $c \geq 1$ and $a \in \mathbb{R}$.)
3. (2 p.) Show that the support of $f_Y(\cdot)$ is $(-1, \infty)$.
4. (3 p.) Let $\alpha \in (0, 1)$. Find constants c_1, c_2 such that $\mathbb{P}(Y < c_1) = \alpha$ and $\mathbb{P}(Y > c_2) = \alpha$. What is the probability $\mathbb{P}(Y \in [c_1, c_2])$?
5. (4 p.) Use the asymptotic distribution of $\frac{\hat{X}_n - \mathbb{E}(\hat{X}_n)}{\sqrt{Var(\hat{X}_n)}}$ to construct an asymptotic random confidence interval for θ with confidence $1 - \alpha$ of the form (C, ∞) .
6. (4 p.) Define a formal hypothesis test with significance α for testing $(H_0) : \theta \geq \theta_0$ against $(H_1) : \theta < \theta_0$. Describe the rejection region implicitly.
7. (1 p.) Determine now the asymptotic rejection region using knowledge from (2) and (5).
8. (3 p.) Assume we get from our data $t(\mathbf{x}) = \ln(10) - 1$. What is the smallest level of significance to reject H_0 in this case? How do we call this value? (Hint: You can use the asymptotic distribution of the test statistic).

Exercise 4 (25 p.) Let X_1, \dots, X_n be an i.i.d. random sample on some probability space with common density

$$f_\theta(x_1) = \frac{x_1}{\theta^2} e^{-\frac{x_1^2}{2\theta^2}}, \quad x_1 \geq 0,$$

and $\theta > 0$. We want to compare two hypothesis tests for $(H_0) : \theta = 1$ against $(H_1) : \theta \neq 1$ using two different test statistics, namely

$$T_1(\mathbf{X}) = \sum_{i=1}^n X_i^2, \quad \text{and} \quad T_2(\mathbf{X}) = 2 \ln(R(\mathbf{X})),$$

where $R(\mathbf{X})$ is the generalized likelihood ratio. You can use that the MLE is equal to $\hat{\theta}_n = \sqrt{\frac{1}{2n} \sum_{i=1}^n X_i^2}$ and is asymptotically normal.

1. (3 p.) Define a formal hypothesis test based on statistic $T_1(\mathbf{X})$ at significance α .
2. (2 p.) Show that the MGF of X_1^2 is equal to $\frac{1}{1-2t\theta^2}$ for $|t| < \frac{1}{2}$.
3. (3 p.) Determine the distribution of $T_1(\mathbf{X})$ under H_0 and show that it equal to $\frac{1}{\theta^2} T_1(\mathbf{X})$ under H_1 .
4. (4 p.) Let $\alpha = 0.05$ and $n = 60$ determine the critical region for this test. Compute the power when $\theta = \sqrt{2}$. If you did not find the distribution before use the normal distribution $N(\theta, 1)$.
5. (4 p.) Consider now the second test statistic $T_2(\mathbf{X})$. Show that $T_2(\mathbf{X})$ is of the form $c_1 \ln(c_2 \sum_{i=1}^n X_i^2) + c_3 \sum_{i=1}^n (X_i^2 - c_4)$ where c_1, c_2, c_3, c_4 are some constants.
6. (4 p.) What is the asymptotic distribution of $T_2(\mathbf{X})$? Determine the asymptotic rejection region for $\alpha = 0.05$.
7. (5 p.) Assume we observe $\sum_{i=1}^{60} x_i^2 = 85$. Perform both tests at $\alpha = 0.05$ and decide whether to reject H_0 or not (you can use the second asymptotic test). If the null hypothesis would be 1-sided, which test is more powerful? Argue without computing the power of the test based on $T_2(\mathbf{X})$.