Tentamen Inleiding Topologie, WISB243 2020-01-28, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 48.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathbb{N} = \{1, 2, 3, \ldots,\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathscr{B}$ be the family of subsets of $\mathbb{N}$ of the form $U_{a,b} = \{a + nb : n \in \mathbb{N}_0\}$ where $a, b \in \mathbb{N}$ are coprime, i.e. $\gcd(a,b) = 1$.

5 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{N}$.

5 pt (b) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ generated by the topology basis $\mathscr{B}$. Show that the topological space $(\mathbb{N}, \mathscr{T})$ is Hausdorff.

9 pt total **Exercise 2.** Let $(X, \mathscr{T})$ be a locally compact Hausdorff space.

3 pt (a) Let $U \subset X$ be an open subset. Show that U is locally compact (with the induced subspace topology).

3 pt (b) Let $C \subset X$ be closed. Show that C is locally compact (with the induced subspace topology).

3 pt (c) Let $U \subset X$ open and $C \subset X$ closed. Show that $U \cap C$ is locally compact.

11 pt total **Exercise 3.** Let $J = [-1, 1]$, let $S := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1, -1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1, z_2, t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t - 1), \quad (0 \leq s, t \leq 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \ \longrightarrow \ X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cylinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any justification.)

10 pt total **Exercise 4.** We call a topological space extremally disconnected space if for every two disjoint open subsets U and V one has $\overline{U} \cap \overline{V} = \emptyset$. Here $\overline{U}$ and $\overline{V}$ are the closures of the sets U and V.

4 pt (a) Prove the following statement: A topological space is extremally disconnected if and only if the closure of every open subset U is open.

3 pt (b) Show that an extremally disconnected Hausdorff space with at least two elements is not connected.

3 pt (c) Give an example of a topological space X that has the following properties

- X has at least two points
- X is connected
- X is extremally disconnected
- for every pair of distinct points, each has a neighborhood not containing the other point.

8 pt total **Exercise 5.** Let X be a compact Hausdorff space and $x_0 \in X$.

3 pt (a) Show that $Y = X \setminus \{x_0\}$ is a locally compact Hausdorff space.

5 pt (b) Let Y_∞ be the one-point compactification of Y. Show that Y is homeomorphic to X.