Hertentamen Inleiding Topologie, WISB243 2020-04-14, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The exam grade E will be obtained from your total score T by rounding off $\min(T/4, 10)$ to one decimal accuracy.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathscr{B}$ be the family of subsets of the form

$$\mathscr{B} = \{(a,b) : -\infty < a < b < \infty\} \bigcup \{\mathbb{Q} \cap (a,b) : -\infty < a < b < \infty\}.$$

2 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{R}$.

2 pt (b) Let $\mathscr{T}$ be the topology generated by $\mathscr{B}$. Show that $(\mathbb{R}, \mathscr{T})$ is a Hausdorff space.

2 pt (c) Show that $\mathbb{R} \setminus \mathbb{Q}$ is closed in $(\mathbb{R}, \mathscr{T})$.

2 pt (d) Show that the topological space $(\mathbb{R}, \mathscr{T})$ is not normal.

2 pt (e) Let $F : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{eucl})$ be a continuous map with $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and where $\mathscr{T}_{eucl}$ is the euclidean topology on $\mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

9 pt total **Exercise 2.** Let $X = \{(x,y) : y \geq 0, (x,y) \neq (0,0)\}$. We understand X as a topological space with the induced subspace topology from $X \subset \mathbb{R}^2$.

3 pt (a) Show that the subsets $E = \{(x,0) : x < 0\}$ and $F = \{(x,0) : x > 0\}$ are closed subsets in X.

3 pt (b) Construct explicitly a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \in F$.

3 pt (c) Is X a normal topological space? (give a proof for your answer).

9 pt total **Exercise 3.** Let (X,d_1) and (X,d_2) be two metric spaces and assume that (X,d_1) is a compact topological space (with the topology indued by the metric). Let $f: X \to Y$ be a continuous map. Show that f is uniformly continuous, i.e. that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon.$$

12 pt total **Exercise 4.** Let X,Y be topological spaces. We call a function $f: X \to Y$ proper if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$.

3 pt (a) Let X,Y be topological spaces and $f: X \to Y$ a continuous map. Show that $f(C) \subset Y$ is compact for every $C \subset X$ compact

3 pt (b) Let X,Y be locally compact Hausdorff spaces with one-point compactifications X_∞ and Y_∞. For a map $f: X \to Y$ we define $\hat{f}: X_\infty \to Y_\infty$ by setting $\hat{f}(x) = f(x)$ for $x \in X$ and $\hat{f}(\infty) = \infty$. Show that $\hat{f}: X_\infty \to Y_\infty$ is continuous if and only if the map $f: X \to Y$ is continuous and proper.

3 pt (c) Let X be a locally compact Hausdorff space and $C \subset X$ a closed subset. Show that $C \cup \{\infty\} \subset X_\infty$ is closed. Moreover show that a subset $C' \subset X_\infty$ is closed if and only if C' is compact.

3 pt (d) Let X,Y be locally compact Hausdorff spaces and $f: X \to Y$ a continuous and proper map. Use the previous parts of this exercise to show that if $C \subset X$ is closed, then $f(C) \subset Y$ is closed.