(1) Consider a 2-period binomial model with \( S_0 = 200 \), \( u = 1.1 \), \( d = 0.8 \), and \( r = 0.05 \). Suppose the real probability measure \( \mathbb{P} \) satisfies \( \mathbb{P}(H) = p = \frac{1}{2} = 1 - \mathbb{P}(T) \). Consider an American put option with expiration date \( N = 2 \) and strike price \( K = 180 \).

(a) Determine the option price process \( (V_n : n = 0, 1, 2) \), and the optimal stopping time \( \tau^* = \inf \{ n \geq 0 : V_n = G_n \} \). (1 pt)

(b) Suppose it is possible to buy or sell the above American put option for a price \( C > V_0 \), where \( V_0 \) is your answer in part (a). Construct an explicit arbitrage strategy assuming that \( \omega_1 \omega_2 = TT \). (1.5 pt)

(c) Consider the stopping time \( \tau \) defined by \( \tau(HH) = \infty \), \( \tau(TH) = \tau(HT) = \tau(TT) = 2 \). Find the value of \( \mathbb{E}\left[ \sum_{n=0}^{2} \mathbb{1}_{\{\tau \leq n\}} \frac{G_n}{(1.05)^n} \right] \). (0.5 pt)

(d) Consider the utility function \( U(x) = \ln x^2 \), \( x > 0 \). Determine explicitly a random variable \( X = X_0 \) (so find \( X(HH), X(HT), X(TT), X(TT) \)) that maximizes \( \mathbb{E}[U(X)] \) subject to the condition that \( X = 200 = \mathbb{E}\left[ \frac{X}{(1.05)^2} \right] \). (1 pt)

Solution(a): We first calculate the risk neutral probability, we have \( \tilde{\mathbb{P}}(H) = \tilde{p} = \frac{5}{6} \) and \( \tilde{q} = \frac{1}{6} \). The intrinsic value process is given by \( G_n = K - S_n = 180 - S_n \). Hence, \( G_0 = -20 \), \( G_1(H) = -40 \), \( G_1(T) = 20 \), \( G_2(HH) = -62 \), \( G_2(HT) = G_2(TH) = 4 \), \( G_2(TT) = 52 \).

Since \( V_2 = \max(G_2, 0) \), we have
\[
V_2(HH) = 0, \quad V_2(HT) = V_2(TH) = 4, \quad V_2(TT) = 52.
\]

By the American algorithm, we have
\[
V_1(H) = \max \left( -40, \frac{5}{6} \right) = \max \left( -40, 0.635 \right) = 0.635
\]
\[
V_1(T) = \max \left( 20, \frac{5}{6} \right) = \max \left( 20, 11.429 \right) = 20
\]
\[
V_0 = \max \left( -20, \frac{5}{6} \right) = \max \left( -20, 0.635 \right) = 0.635
\]

From the above calculations, we see that \( \tau^*(HH) = \infty \), \( \tau^*(HT) = 2 \), \( \tau^*(TH) = \tau^*(TT) = 1 \).

Solution(b): We perform the following arbitrage strategy:

-at time 0:
  - We sell the option for \( C \) euros
  - Deposit \( C - V_0 = C - 3.679 \) in the bank
  - Use \( V_0 \) to start a self-financing wealth process, so amount of stocks is
\[
\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{0.635 - 20}{220 - 160} = -0.32275.
\]

So sell 0.32275 of a share and deposit \( V_0 - \Delta_0 S_0 = 68.229 \) in the money market, so \( X_0 = V_0 = \Delta_0 S_0 + (V_0 - \Delta_0 S_0) \).
Consider the binomial model with up factor 

\[ u = 2, \text{ down factor } d = 1/2 \]

and interest rate \( r = 1/4 \). Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 1 euro. What is then the price at time 0 of such an option? (0.5 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price rises to 8 euros. What is then the price at time 0 of such an option? (0.5 pt)

(c) Determine under \( \mathbb{P} \), the probability that the price reaches 8 euros for the first time at time \( n = 3 \)? (0.5)
(d) Consider the process \( v(S_0), v(S_1), \ldots \) defined by
\[
v(S_n) = \begin{cases} 
16 - S_n, & \text{if } S_n \leq 8, \\
64/S_n, & \text{if } S_n \geq 8.
\end{cases}
\]
Show that the discounted process \( \{ (4/5)^n v(S_n) : n = 0, 1, \ldots \} \) is a supermartingale under \( \tilde{\mathbb{P}} \). (1.5 pt)

Solution (a): The buyer is using the exercise policy \( \tau_{-2} \). Hence, the price at time 0 should be
\[
V_0 = V^{\tau_{-2}} = \tilde{\mathbb{E}} \left( \left( \frac{4}{5} \right)^{\tau_{-2}} (16 - S_{\tau_{-2}}) \right) \\
= \left( \frac{1}{2} \right)^2 (16 - 1) = \frac{15}{4} = 3.75.
\]

Solution (b): The buyer is using the exercise policy \( \tau_1 \). Hence, the price at time 0 should be
\[
V_0 = V^{\tau_1} = \tilde{\mathbb{E}} \left( \left( \frac{4}{5} \right)^{\tau_1} (16 - S_{\tau_1}) \right) \\
= \left( \frac{1}{2} \right)^1 (16 - 8) = \frac{8}{2} = 4.
\]

Solution (c): The probability that the price reaches 8 for the first time at time 3 is equal to the \( \mathbb{P}(\{\tau_1 = 3\}) \). By Theorem 5.2.5,
\[
\mathbb{P}(\{\tau_1 = 3\}) = \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right) = \frac{4}{27}.
\]

Solution (d): If \( S_n < 8 \), then \( S_{n+1} \leq 8 \), thus
\[
\tilde{\mathbb{E}}_n \left( \left( \frac{4}{5} \right)^{n+1} v(S_{n+1}) \right) = \left( \frac{4}{5} \right)^{n+1} \left[ \frac{1}{2} (16 - 2S_n) + \frac{1}{2} (16 - S_n/2) \right] \\
= \left( \frac{4}{5} \right)^{n} \left( \frac{4}{5} 16 - S_n \right) \\
< \left( \frac{4}{5} \right)^{n} (16 - S_n) = \left( \frac{4}{5} \right)^{n} v(S_n).
\]
If \( S_n = 8 \), then \( S_{n+1} \in \{4, 16\} \). Thus,
\[
\tilde{\mathbb{E}}_n \left( \left( \frac{4}{5} \right)^{n+1} v(S_{n+1}) \right) = \left( \frac{4}{5} \right)^{n+1} \left[ \frac{1}{2} (12) + \frac{1}{2} (4) \right] \\
= \left( \frac{4}{5} \right)^{n} \left( \frac{4}{5} 8 \right) \\
< \left( \frac{4}{5} \right)^{n} \ 8 \\
= \left( \frac{4}{5} \right)^{n} v(S_n).
\]
If \( S_n \geq 16 \), then
\[
\tilde{\mathbb{E}}_n \left( \left( \frac{4}{5} \right)^{n+1} v(S_{n+1}) \right) = \left( \frac{4}{5} \right)^{n+1} \left[ \frac{1}{2} \frac{64}{S_n} + \frac{1}{2} \frac{128}{S_n} \right] \\
= \left( \frac{4}{5} \right)^{n} \frac{64}{S_n} = \left( \frac{4}{5} \right)^{n} v(S_n).
\]
In all cases we have \( \tilde{\mathbb{E}}_n \left( (\frac{4}{5})^{n+1} v(S_{n+1}) \right) \leq (\frac{4}{5})^n v(S_n) \) as required.
(3) Consider the (infinite) binomial model with \( \mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} \), so the underlying space is given by \( \Omega = \{ (\omega_1, \omega_2, \cdots) : \omega_i \in \{H,T\} \} \). Define
\[
X_n = \begin{cases} 
1, & \text{if } \omega_n = H, \\
-1, & \text{if } \omega_n = T,
\end{cases}
\]
for \( n = 1, 2, \cdots \). Let \( U_0 = 0 \) and
\[
U_n = \sum_{k=1}^{n} 2^{k-1} X_k = X_1 + 2X_2 + 2^2 X_3 + \cdots + 2^{n-1} X_n, \]
for \( n = 1, 2, \cdots \). Define the stopping time \( \tau \) by \( \Omega = \{ \omega \} \) with
\[
2^{\tau-1} \leq X_\tau < 2^{\tau},
\]
for \( \tau = 1, \) otherwise.

\( \tau = 1 \) and \( X_\tau = -1 \) contradicting the definition of \( \tau \). We claim that \( \tau = n \). Assume not, i.e. \( 2 \leq j < n \), then \( X_1 = X_2 = \cdots = X_{j-1} = -1 \) and \( X_j = 1 \). Then,
\[
U_j = \sum_{i=1}^{j} 2^{i-1} X_i = -(2^{j-2} + \cdots + 1) + 2^{j-1} = -(2^{j-1} - 1) + 2^{j-1} = 1,
\]
contradicting the fact that \( n \) is the first index such that \( U_n = 1 \). Thus \( j = n \) and \( X_1 = \cdots = X_{n-1} = -1 \) and \( X_n = 1 \). This shows that \( \mathbb{P}(\{ \tau = n \}) = \mathbb{P}(\{ X_1 = -1, \cdots, X_{n-1} = -1, X_n = 1 \}) \).