Uitwerkingen Deeltentamen 2: Inleiding Finaniele Wiskunde 2017-2018

(1) Consider a 2-period binomial model with $S_0 = 200$, $u = 1.1$, $d = 0.8$, and $r = 0.05$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = 1 - \mathbb{P}(T)$. Consider an American put option with expiration date $N = 2$ and strike price $K = 180$.

(a) Determine the option price process $(V_n : n = 0, 1, 2)$, and the optimal stopping time $\tau^* = \inf\{n \geq 0 : V_n = G_n\}$. (1 pt)

(b) Suppose it is possible to buy or sell the above American put option for a price $C > V_0$, where V_0 is your answer in part (a). Construct an explicit arbitrage strategy assuming that $\omega_1\omega_2 = TT$. (1.5 pt)

(c) Consider the stopping time τ defined by $\tau(HH) = \infty$, $\tau(TH) = \tau(HT) = \tau(TT) = 2$. Find the value of $\widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau\leq 2\}} \frac{G_\tau}{(1.05)^\tau}\Big]$. (0.5 pt)

(d) Consider the utility function $U(x) = \ln x^2$, $x > 0$. Determine explicitly a random variable $X = X_2$ (so find $X(HH), X(HT), X(TH), X(TT)$) that maximizes $\mathbb{E}[U(X)]$ subject to the condition that $X_0 = 200 = \widetilde{\mathbb{E}}\Big[\frac{X}{(1.05)^2}\Big]$. (1 pt)

Solution(a): We first calculate the risk neutral probability, we have $\widetilde{\mathbb{P}}(H) = \widetilde{p} = \frac{5}{6}$ and $\widetilde{q} = \frac{1}{6}$. The intrinsic value process is given by $G_n = K - S_n = 180 - S_n$. Hence,

$$G_0 = -20,\ G_1(H) = -40,\ G_1(T) = 20,\ G_2(HH) = -62,\ G_2(HT) = G_2(TH) = 4,\ G_2(TT) = 52.$$

Since $V_2 = \max(G_2, 0)$, we have

$$V_2(HH) = 0,\ V_2(HT) = V_2(TH) = 4,\ V_2(TT) = 52.$$

By the American algorithm, we have

$$V_1(H) = \max\Big(-40, \frac{1}{1.05}[\frac{5}{6}(0) + \frac{1}{6}(4)]\Big) = \max\Big(-40, 0.635\Big) = 0.635$$
$$V_1(T) = \max\Big(20, \frac{1}{1.05}[\frac{5}{6}(4) + \frac{1}{6}(52)]\Big) = \max\Big(20, 11.429\Big) = 20$$
$$V_0 = \max\Big(-20, \frac{1}{1.05}[\frac{5}{6}(0.635) + \frac{1}{6}(20)]\Big) = \max\Big(-20, 3.679\Big) = 3.679.$$

From the above calculations, we see that

$$\tau^*(HH) = \infty,\ \tau^*(HT) = 2,\ \tau^*(TH) = \tau^*(TT) = 1.$$

Solution(b): We perform the following arbitrage strategy:

–at time 0:
- We sell the option for C euros
- Deposit $C - V_0 = C - 3.679$ in the bank
- Use V_0 to start a self-financing wealth process, so amount of stocks is

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{0.635 - 20}{220 - 160} = -0.32275.$$

So sell 0.32275 of a share and deposit $V_0 - \Delta_0 S_0 = 68.229$ in the money market, so $X_0 = V_0 = \Delta_0 S_0 + (V_0 - \Delta_0 S_0)$.

–at time 1, we have $\omega_1 = T$ and the wealth process has value

$$X_1(T) = \Delta_0 S_1(T) + (1.05)(V_0 - \Delta_0 S_0) = 20 = V_1(T).$$

- If the buyer exercises, you pay him the $X_1(T) = 20$, and the deal is closed. Your bank account has grown to $(1.05)(C - 3.679) > 0$, so you end up with a positive profit.

- If the buyer does not exercise, then you adjust your portfolio to

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = \frac{4 - 52}{176 - 128} = -1.$$

  So you sell 1 share and you are allowed to consume $C_1(T) = 20 - 11.429 = 8.571$ which you deposit in the bank. At time 2, $\omega_2 = T$ and your wealth process has value

$$X_2(TT) = \Delta_1(T)S_2(TT) + (1.05)(X_1(T) - C_1(T) - \Delta_1(T)S_1(T)) = 52 = V_2(TT).$$

  In this case, the buyer of the option exercises, so you pay him $X_2(TT) = 52$, and your bank account has grown to $(1.05)^2(C-3.679)+(1.05)(8.571) > 0$, ending up with a positive profit.

Solution(c):

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau\le 2\}}\frac{G_\tau}{(1.05)^\tau}\Big] &= \frac{1}{(1.05)^2}\Big[G_2(HT)\widetilde{\mathbb{P}}(HT) + G_2(TH)\widetilde{\mathbb{P}}(TH) + G_2(TT)\widetilde{\mathbb{P}}(TT)\Big] \\
&= \frac{1}{(1.05)^2}\Big[(4)(\frac{5}{6})(\frac{1}{6}) + (4)(\frac{5}{6})(\frac{1}{6}) + (52)(\frac{1}{6})(\frac{1}{6})\Big] \\
&= 2.318.
\end{aligned}$$

Solution(d): Notice that the function $U(x) = \ln x^2 = 2\ln x$, $x > 0$ is strict concave with $U'(x) = \frac{2}{x}$. We apply Theorem 3.3.6, we find that the inverse I of U' is given by $I(x) = \frac{2}{x} = U'(x)$. Thus, the optimal solution is given by

$$X_2 = X = I\left(\frac{\lambda Z}{(1.05)^2}\right) = \frac{2(1.05)^2}{\lambda Z},$$

and satisfies the constraint

$$X_0 = 200 = \mathbb{E}\left(\frac{XZ}{(1.05)^2}\right) = \frac{2}{\lambda}.$$

Hence, $\lambda = \frac{1}{100}$ and $X = \frac{220.5}{Z}$. To find the explicit value of X, we need to calculate the values of the Radon-Nikodym derivative Z. First, we see that $\frac{\widetilde{p}}{p} = \frac{5}{3}$ and $\frac{\widetilde{q}}{q} = \frac{1}{3}$. Hence,

$$Z(HH) = \frac{25}{9} = 2.78,\ Z(HT) = Z(TH) = \frac{5}{9} = 0.56,\ Z(TT) = \frac{1}{9} = 0.11.$$

Therefore,

$$X(HH) = 79.38,\ X(HT) = X(TH) = 396.9,\ X(TT) = 1984.5.$$

(2) Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$ and a real probability $\mathbb{P}$ given by $\mathbb{P}(H) = 2/3$ and $\mathbb{P}(T) = 1/3$. Consider a perpetual American put option with $S_0 = 4$ and strike price $K = 16$.

  (a) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 1 euro. What is then the price at time 0 of such an option? (0.5 pt)

  (b) Suppose the buyer of the option uses the strategy of exercising the first time the price rises to 8 euros. What is then the price at time 0 of such an option? (0.5 pt)

  (c) Determine under $\mathbb{P}$, the probability that the price reaches 8 euros for the first time at time $n = 3$? (0.5)

(d) Consider the process $v(S_0), v(S_1), \cdots$ defined by

$$v(S_n) = \begin{cases} 16 - S_n, & \text{if } S_n \leq 8, \\ \dfrac{64}{S_n}, & \text{if } S_n \geq 8. \end{cases}$$

Show that the discounted process $\{\left(\frac{4}{5}\right)^n v(S_n) : n = 0, 1, \cdots\}$ is a supermartingale under $\widetilde{\mathbb{P}}$. (1.5 pt)

Solution (a): The buyer is using the exercise policy τ_{-2}. Hence, the price at time 0 should be

$$\begin{aligned} V_0 = V^{\tau_{-2}} &= \widetilde{\mathbb{E}}\left(\left(\frac{4}{5}\right)^{\tau_{-2}} (16 - S_{\tau_{-2}})\right) \\ &= (\frac{1}{2})^2(16-1) = \frac{15}{4} = 3.75. \end{aligned}$$

Solution (b): The buyer is using the exercise policy τ_1. Hence, the price at time 0 should be

$$\begin{aligned} V_0 = V^{\tau_1} &= \widetilde{\mathbb{E}}\left(\left(\frac{4}{5}\right)^{\tau_1} (16 - S_{\tau_1})\right) \\ &= (\frac{1}{2})^1(16-8) = \frac{8}{2} = 4. \end{aligned}$$

Solution (c): The probability that the price reaches 8 for the first time at time 3 is equal to the $\mathbb{P}(\{\tau_1 = 3\})$. By Theorem 5.2.5,

$$\mathbb{P}(\{\tau_1 = 3\}) = (\frac{2}{3})^2(\frac{1}{3}) = \frac{4}{27}.$$

Solution (d): If $S_n < 8$, then $S_{n+1} \leq 8$, thus

$$\begin{aligned} \widetilde{\mathbb{E}}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}(16 - 2S_n) + \frac{1}{2}(16 - S_n/2)] \\ &= \left(\frac{4}{5}\right)^{n} (\frac{4}{5}16 - S_n) \\ &< \left(\frac{4}{5}\right)^{n} (16 - S_n) = \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

If $S_n = 8$, then $S_{n+1} \in \{4, 16\}$. Thus,

$$\begin{aligned} \widetilde{\mathbb{E}}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}(12) + \frac{1}{2}(4)] \\ &= \left(\frac{4}{5}\right)^{n} (\frac{4}{5})(8) \\ &< \left(\frac{4}{5}\right)^{n} 8 \\ &= \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

If $S_n \geq 16$, then

$$\begin{aligned} \widetilde{\mathbb{E}}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}\frac{64}{2S_n} + \frac{1}{2}\frac{128}{S_n}] \\ &= \left(\frac{4}{5}\right)^{n} \frac{64}{S_n} = \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

In all cases we have $\widetilde{\mathbb{E}}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) \leq \left(\frac{4}{5}\right)^n v(S_n)$ as required.

(3) Consider the (infinite) binomial model with $\mathbb{P}(H) = \mathbb{P}(T) = \dfrac{1}{2}$, so the underlying space is given by $\Omega = \{(\omega_1, \omega_2, \cdots) : \omega_i \in \{H, T\}\}$. Define

$$X_n = \begin{cases} 1, & \text{if } \omega_n = H, \\ -1, & \text{if } \omega_n = T, \end{cases}$$

$n = 1, 2, \cdots$. Let $U_0 = 0$ and

$$U_n = \sum_{k=1}^{n} 2^{k-1} X_k = X_1 + 2X_2 + 2^2 X_3 + \cdots + 2^{n-1} X_n,$$

$n = 1, 2, \cdots$. Define the stopping time τ by $\tau = \inf\{n \geq 1 : U_n = 1\}$.

(a) Prove that the process $(U_n : n = 0, 1, \cdots)$ is a martingale (under the usual filtration $\mathcal{F}_n$, the information of the first n coin flips). (1 pt)

(b) Prove that $\mathbb{E}[U_{n\wedge\tau}] = 0$ for all $n = 0, 1, \cdots$. (0.5 pt)

(c) Prove that $\mathbb{P}(\{\tau = n\}) = \mathbb{P}(\{X_1 = -1, \cdots, X_{n-1} = -1, X_n = 1\})$, for $n = 2, 3, \cdots$. What is the value of $\mathbb{P}(\{\tau = 1\})$? (Hint $\sum_{k=0}^{n-1} 2^k = 2^n - 1$). (1.5 pt)

Solution (a): Clearly the process $(U_n : n = 0, 1, \cdots)$ is adapted, and for $n \geq 0$,

$$\begin{aligned} \mathbb{E}_n[U_{n+1}] &= \mathbb{E}_n\Big[U_n + 2^n X_{n+1}\Big] \\ &= U_n + 2^n \mathbb{E}_n[X_{n+1}] \\ &= U_n + 2^n \mathbb{E}[X_{n+1}] \\ &= U_n, \end{aligned}$$

where the second equality follows from the linearity of the conditional expectation and that U_n is known at time n, the third and fourth equalities follow from the fact that X_{n+1} is independent from the first n tosses so that $\mathbb{E}_n[X_{n+1}] = \mathbb{E}[X_{n+1}] = 0$.

Solution (b): By the Optional Sampling: Part I, the process $(U_{n\wedge\tau} : n = 0, 1, \cdots)$ is a martingale, hence (one can also apply the Optional Sampling: Part II directly)

$$\mathbb{E}[U_{n\wedge\tau}] = \mathbb{E}[U_{0\wedge\tau}] = \mathbb{E}[U_0] = 0.$$

Solution (c): Since $U_0 = 0$, then $\mathbb{P}(\{\tau = 1\}) = \mathbb{P}(\{X_1 = 1\}) = \dfrac{1}{2}$. Let $n \geq 2$ and consider the event $\{\tau = n\}$. Since $n \geq 2$, then $X_1 = -1$ and there must exist a **least** $j \leq n$ such that $X_j = 1$, otherwise $U_n < 0$ contradicting the definition of τ. We claim that $j = n$. Assume not, i.e. $2 \leq j < n$, then $X_1 = X_2 = \cdots = X_{j-1} = -1$ and $X_j = 1$. Then,

$$U_j = \sum_{i=1}^{j} 2^{i-1} X_i = -(2^{j-2} + \cdots + 1) + 2^{j-1} = -(2^{j-1} - 1) + 2^{j-1} = 1,$$

contradicting the fact that n is the first index such that $U_n = 1$. Thus $j = n$ and $X_1 = \cdots = X_{n-1} = -1$ and $X_n = 1$. This shows that $\mathbb{P}(\{\tau = n\}) = \mathbb{P}(\{X_1 = -1, \cdots, X_{n-1} = -1, X_n = 1\})$.