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Solutions Final: Inleiding Financiele Wiskunde 2018-2019

(1) Consider a Brownian motion $\{W(t) : t \geq 0\}$ with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Suppose that the price process $\{S(t) : t \geq 0\}$ of a certain stock is modelled as the following Itô-process

$$S(t) = S(0) + \int_0^t \mu S(u)\, du + \int_0^t \sigma\, dW(u).$$

(a) Use Itô-Doeblin formula to show that $e^{-\mu t} S(t) = S(0) + \int_0^t e^{-\mu u} \sigma\, dW(u)$. (1 pt)

(b) Determine the distribution of $S(t)$ and calculate $\mathbb{P}(S(t) < 0)$ for $t > 0$. (1 pt)

Proof (a) : First observe that

$$d(S(t)) = \mu\, S(t)\, dt + \sigma\, dW(t).$$

Consider the function $f(t,x) = e^{-\mu t}x$, clearly f has continuous first and second partial derivatives. We have $f_t(t,x) = -\mu e^{-\mu t}x$, $f_x(t,x) = e^{-\mu t}$ and $f_{xx}(t,x) = 0$. By Itô-Doeblin formula, we get

$$\begin{aligned} e^{-\mu t}S(t) &= S(0) + \int_0^t -\mu e^{-\mu u} S(u)\, du + \int_0^t e^{-\mu u}\, dS(u) \\ &= S(0) + \int_0^t -\mu e^{-\mu u} S(u)\, du + \int_0^t e^{-\mu u}\, (\mu\, S(u)\, du + \sigma\, dW(u)) \\ &= S(0) + \int_0^t e^{-\mu u} \sigma\, dW(u) \end{aligned}$$

Proof (b) : From part (a), we have $S(t) = S(0)e^{\mu t} + \int_0^t e^{\mu(t-u)} \sigma\, dW(u)$. Note that $\int_0^t e^{\mu(t-u)}\sigma\, dW(u)$ is an Itô-integral of a deterministic process, hence it is normally distributed with mean 0 and variance

$$\mathrm{Var}\Big(\int_0^t e^{\mu(t-u)} \sigma\, dW(u)\Big) = \sigma^2 e^{2\mu t} \int_0^t e^{-2\mu u}\, du = \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1).$$

From this it follows that $S(t)$ is normally distributed with $\mathbb{E}(S(t)) = S(0)e^{\mu t}$ and $\mathrm{Var}(S(t)) = \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$. Finally, we have

$$\mathbb{P}(S(t) < 0) = \mathbb{P}\Big(\frac{S(t) - S(0)e^{\mu t}}{\sqrt{\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)}} < \frac{-S(0)e^{\mu t}}{\sqrt{\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)}}\Big) = N\Big(\frac{-S(0)e^{\mu t}}{\sqrt{\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)}},\Big)$$

where $N(x)$ is the standard normal distribution function.

(2) Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by

$$\begin{aligned} dS_1(t) &= \alpha S_1(t)\, dW_1(t) + \beta S_1(t)\, dW_2(t) \\ dS_2(t) &= \gamma S_2(t)\, dt + \sigma S_2(t)\, dW_1(t), \end{aligned}$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

(a) Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (1 pt)

(b) Show that $\mathbb{E}[S_1(t)S_2(t)] = S_1(0)S_2(0)e^{(\gamma+\alpha\sigma)t}$, $t \geq 0$. (You are allowed to interchange integrals and expectations). (1 pt)

(c) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\widetilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (1.5 pt)

Proof (a) : We apply Itô product rule, we have

$$d(S_1(t)S_2(t)) = S_1(t)\,dS_2(t) + S_2(t)\,dS_1(t) + dS_1(t)\,dS_2(t).$$

Using $dS_1(t) = \alpha S_1(t)\,dW_1(t) + \beta S_1(t)\,dW_2(t)$, $dS_2(t) = \gamma S_2(t)\,dt + \sigma S_2(t)\,dW_1(t)$ and simplifying, we get

$$d(S_1(t)S_2(t) = (\gamma + \alpha\sigma)S_1(t)S_2(t)\,dt + (\sigma + \alpha)S_1(t)S_2(t)\,dW_1(t) + \beta S_1(t)S_2(t)\,dW_2(t).$$

Equivalently,

$$\begin{aligned} S_1(t)S_2(t) = S_1(0)S_2(0) + \int_0^t (\gamma + \alpha\sigma)S_1(u)S_2(u)\,du + \int_0^t (\sigma + \alpha)S_1(u)S_2(u)\,dW_1(u) \\ + \int_0^t \beta S_1(u)S_2(u)\,dW_2(u). \end{aligned}$$

Hence, $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô process.

Proof (b) : Since Itô-integrals have zero expectation, we have

$$\begin{aligned} \mathbb{E}[S_1(t)S_2(t)] &= S_1(0)S_2(0) + \mathbb{E}\Big[\int_0^t (\gamma + \alpha\sigma)S_1(u)S_2(u)\,du\Big] \\ &= S_1(0)S_2(0) + \int_0^t (\gamma + \alpha\sigma)\mathbb{E}[S_1(u)S_2(u)]\,du. \end{aligned}$$

Let $m(t) = \mathbb{E}[S_1(t)S_2(t)]$, then the above equation reads

$$m(t) = m(0) + \int_0^t (\gamma + \alpha\sigma)m(u)\,du,$$

which upon differentiation gives

$$\frac{dm(t)}{dt} = (\gamma + \alpha\sigma)m(t).$$

The latter has solution $m(t) = m(0)e^{(\gamma+\alpha\sigma)t}$. Hence, $\mathbb{E}[S_1(t)S_2(t)] = S_1(0)S_2(0)e^{(\gamma+\alpha\sigma)t}$, $t \geq 0$.

Proof (c) : Using the notation of the book, we have $\alpha_1 = 0$, $\sigma_{11} = \alpha$, $\sigma_{12} = \beta$, $\alpha_2 = \gamma$, $\sigma_{21} = \sigma$, $\sigma_{22} = 0$. The market price equations in this case is the system,

$$\begin{aligned} -r &= \alpha\theta_1(t) + \beta\theta_2(t) \\ \gamma - r &= \sigma\theta_1(t). \end{aligned}$$

Solving for $\theta_1(t), \theta_2(t)$, we get

$$\begin{aligned} \theta_1(t) &= \frac{\gamma - r}{\sigma} \\ \theta_2(t) &= -\frac{\sigma r + \alpha(\gamma - r)}{\sigma\beta}. \end{aligned}$$

Setting

$$\begin{aligned} Z &= \exp\Big\{ -\int_0^T \Big(\theta_1(t)\,dW_1(t) + \theta_2(t)\,dW_2(t)\Big) - \frac{1}{2}\int_0^T \Big(\theta_1^2(t) + \theta_2^2(t)\Big)\,dt\Big\} \\ &= \exp\Big\{ -\frac{\gamma - r}{\sigma}W_1(T) + \frac{\sigma r + \alpha(\gamma - r)}{\sigma\beta}W_2(T) - \frac{1}{2}\Big(\frac{(\gamma - r)^2}{\sigma^2} + \frac{(\sigma r + \alpha(\gamma - r))^2}{\sigma^2\beta^2}\Big)T\Big\}, \end{aligned}$$

the risk-neutral measure is given by $\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P}$. To check this, we set $\widetilde{W}_1(t) = \frac{\gamma-r}{\sigma}t + W_1(t)$ and $\widetilde{W}_2(t) = -\frac{\sigma r+\alpha(\gamma-r)}{\sigma\beta}t + W_2(t)$, By the 2-dimensional Girsanov Theorem the process $\{(\widetilde{W}_1(t), \widetilde{W}_2(t) : 0 \le t \le T\}$ is a 2-dimensional Brownian motion under $\widetilde{\mathbb{P}}$. Rewriting $e^{-rt}S_1(t), e^{-rt}S_2(t)$ in terms of $\widetilde{W}_1(t), \widetilde{W}_1(t)$, we get after applying Itô product rule

$$d(e^{-rt}S_1(t)) = e^{-rt}S_1(t)(\alpha d\widetilde{W}_1(t) + \beta d\widetilde{W}_2(t))$$
$$d(e^{-rt}S_2(t)) = e^{-rt}S_2(t)\sigma d\widetilde{W}_1(t),$$

which shows that the discounted price processes are Itô integrals and hence martingales under $\widetilde{\mathbb{P}}$.

(3) Let T fe finite horizon and let $\{W(t) : 0 \le t \le T\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}(t) : 0 \le t \le T\}$, where $\mathcal{F}(T) = \mathcal{F}$. Suppose that the price process $\{S(t) : 0 \le t \le T\}$ of a certain stock is given by

$$S(t) = \exp\left\{2W(t) + \frac{t^2}{2} - 2t\right\}$$

(a) Show that $\{S(t) : 0 \le t \le T\}$ is an Itô-process. (1 pt)

(b) Let r be a constant interest rate. Find a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted process $\{e^{-rt}S(t) : 0 \le t \le T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1 pt)

Proof (a): We apply Itô-Doeblin to the function $f(t,x) = e^{2x+\frac{t^2}{2}-2t}$. We first calculate the partial derivatives, we have $f_t(t,x) = (t-2)f(t,x)$, $f_x(t,x) = 2f(t,x)$ and $f_{xx}(t,x) = 4f(t,x)$. Then,

$$\begin{aligned} dS(t) = df(t, W(t)) &= (t-2)S(t)\,dt + 2S(t)\,dW(t) + 2S(t)\,dt \\ &= tS(t)\,dt + 2S(t)\,dW(t). \end{aligned}$$

Hence, $S(t) = S(0) + \int_0^t uS(u)\,du + \int_0^t 2S(u)\,dW(u)$, and therefore $\{S(t) : 0 \le t \le T\}$ is an Itô-process.

Proof (b): We consider the adapted process $\{\theta(t) : 0 \le t \le T\}$ given by $\theta(t) = \frac{t-r}{2}$, and the random variable

$$Z = \exp\Big\{ -\int_0^T \theta(u)\,dW(u) - \frac{1}{2}\int_0^T \theta^2(u)\,du.\Big\}$$

Notice that θ is bounded on the interval $[0,T]$, hence $\mathbb{E}\Big[\int_0^T \theta^2(u)z^2(u)\,du\Big] < \infty$. Consider the probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ defined by $\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P}$, and the process $\{\widetilde{W}(t) : 0 \le t \le T\}$ with $\widetilde{W}(t) = \int_0^t \theta(u)\,du + W(t)$. By Girsanov's Theorem, the process $\{\widetilde{W}(t) : 0 \le t \le T\}$ is a Brownian motion under $\widetilde{\mathbb{P}}$, and by Itô product rule we have,

$$\begin{aligned} d(e^{-rt}S(t)) &= e^{-rt}\,dS(t) - re^{-rt}S(t)\,dt \\ &= e^{-rt}[tS(t)\,dt + 2S(t)\,dW(t)] - re^{-et}S(t)\,dt \\ &= e^{-rt}(t-r)S(t)\,dt + 2e^{-rt}S(t)\,dW(t) \\ &= e^{-rt}2\theta(t)S(t)\,dt + 2e^{-rt}S(t)\,dW(t) \\ &= 2e^{-rt}S(t)\Big(\theta(t)\,dt + dW(t)\Big) \\ &= 2e^{-rt}S(t)d\widetilde{W}(t). \end{aligned}$$

This shows that the process $\{e^{-rt}S(t) : 0 \le t \le T\}$ is an Itô process under $\widetilde{\mathbb{P}}$ and hence a martingale under $\widetilde{\mathbb{P}}$.

(4) Consider a Brownian motion $\{W(t) : t \geq 0\}$ with the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$, where $\mathcal{F}(t) = \sigma(\{W(s) : s \leq t\})$. Consider the stochastic process $\{M(t) : t \geq 0\}$, with

$$M(t) = \Big(\int_0^t sW^2(s)\, dW(s) \Big)^2 - \int_0^t s^2 W^4(s)\, ds.$$

(a) Determine the value of $\mathbb{E}[M(t)]$ for $t \geq 0$. (1 pt)

(b) Prove that the stochastic process $\{M(t) : t \geq 0\}$ is a martingale with respect to the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1.5 pt)

Proof (a) : We use Itô-isometry. For any $t \geq 0$ we have,

$$\begin{aligned}
\mathbb{E}[M(t)] &= \mathbb{E}\Big[\Big(\int_0^t sW^2(s)\, dW(s) \Big)^2\Big] - \mathbb{E}\Big[\int_0^t s^2 W^4(s)\, ds\Big] \\
&= \mathbb{E}\Big[\int_0^t s^2 W^4(s)\, d(s)\Big] - E\Big[\int_0^t s^2 W^4(s)\, ds\Big] \\
&= 0.
\end{aligned}$$

Proof (b) : First note that the process $\{M(t) : t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. To prove that the stochastic process $\{M(t) : t \geq 0\}$ is a martingale with respect to the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$, it is enough to show that $M(t)$ is an Itô-integral. To this end, define $X(t) = \int_0^t sW^2(s)\, dW(s)$ and $Y(t) = \int_0^t s^2 W^4(s)\, ds$. Then $dX(t) = tW^2(t)\, dW(t)$ and $dY(t) = t^2 W^4(t)\, dt$. Consider the function $f(x, y) = x^2 - y$, then $f_x(x, y) = 2x$, $f_{xx}(x, y) = 2$, $f_y(x, y) = -1$ and $f_{yy}(x, y) = 0$. By Itô-Doeblin formula we have

$$\begin{aligned}
dM(t) = df(X(t), Y(t)) &= 2X(t)\, dX(t) - dY(t) + dX(t)\, dX(t) \\
&= 2X(t) t W^2(t)\, dW(t) - t^2 W^4(t)\, dt + t^2 W^4(t)\, dt \\
&= 2X(t) t W^2(t)\, dW(t).
\end{aligned}$$

Equivalently, $M(t) = M(0) + \int_0^t 2X(u) u W^2(u)\, dW(u)$ and therefore $\{M(t) : t \geq 0\}$ is an Itô-process. This proves that the process $\{M(t) : t \geq 0\}$ is a martingale with respect to the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$.