Deeltentamen 2: Inleiding Finaniele Wiskunde 2017-2018

(1) Consider a 2-period binomial model with $S_0 = 200$, $u = 1.1$, $d = 0.8$, and $r = 0.05$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = 1 - \mathbb{P}(T)$. Consider an American put option with expiration date $N = 2$ and strike price $K = 180$.

(a) Determine the option price process $(V_n : n = 0, 1, 2)$, and the optimal stopping time $\tau^* = \inf\{n \geq 0 : V_n = G_n\}$. (1 pt)

(b) Suppose it is possible to buy or sell the above American put option for a price $C > V_0$, where V_0 is your answer in part (a). Construct an explicit arbitrage strategy assuming that $\omega_1\omega_2 = TT$. (1.5 pt)

(c) Consider the stopping time τ defined by $\tau(HH) = \infty$, $\tau(TH) = \tau(HT) = \tau(TT) = 2$. Find the value of $\widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau \leq 2\}} \dfrac{G_\tau}{(1.05)^\tau}\Big]$. (0.5 pt)

(d) Consider the utility function $U(x) = \ln x^2$, $x > 0$. Determine explicitly a random variable $X = X_2$ (so find $X(HH), X(HT), X(TH), X(TT)$) that maximizes $\mathbb{E}[U(X)]$ subject to the condition that $X_0 = 200 = \widetilde{\mathbb{E}}\Big[\dfrac{X}{(1.05)^2}\Big]$. (1 pt)

(2) Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$ and a real probability $\mathbb{P}$ given by $\mathbb{P}(H) = 2/3$ and $\mathbb{P}(T) = 1/3$. Consider a perpetual American put option with $S_0 = 4$ and strike price $K = 16$.

(a) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 1 euro. What is then the price at time 0 of such an option? (0.5 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price rises to 8 euros. What is then the price at time 0 of such an option? (0.5 pt)

(c) Determine under $\mathbb{P}$, the probability that the price reaches 8 euros for the first time at time $n = 3$? (0.5)

(d) Consider the process $v(S_0), v(S_1), \cdots$ defined by

$$v(S_n) = \begin{cases} 16 - S_n, & \text{if } S_n \leq 8, \\ \dfrac{64}{S_n}, & \text{if } S_n \geq 8. \end{cases}$$

Show that the discounted process $\{\left(\frac{4}{5}\right)^n v(S_n) : n = 0, 1, \cdots\}$ is a supermartingale under $\widetilde{\mathbb{P}}$. (1.5 pt)

(3) Consider the (infinite) binomial model with $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$, so the underlying space is given by $\Omega = \{(\omega_1, \omega_2, \cdots) : \omega_i \in \{H, T\}\}$. Define

$$X_n = \begin{cases} 1, & \text{if } \omega_n = H, \\ -1, & \text{if } \omega_n = T, \end{cases}$$

$n = 1, 2, \cdots$. Let $U_0 = 0$ and

$$U_n = \sum_{k=1}^{n} 2^{k-1} X_k = X_1 + 2X_2 + 2^2 X_3 + \cdots + 2^{n-1} X_n,$$

$n = 1, 2, \cdots$. Define the stopping time τ by $\tau = \inf\{n \geq 1 : U_n = 1\}$.

(a) Prove that the process $(U_n : n = 0, 1, \cdots)$ is a martingale (under the usual filtration $\mathcal{F}_n$, the information of the first n coin flips). (1 pt)

(b) Prove that $\mathbb{E}[U_{n\wedge\tau}] = 0$ for all $n = 0, 1, \cdots$. (0.5 pt)

(c) Prove that $\mathbb{P}(\{\tau = n\}) = \mathbb{P}(\{X_1 = -1, \cdots, X_{n-1} = -1, X_n = 1\})$, for $n = 2, 3, \cdots$. What is the value of $\mathbb{P}(\{\tau = 1\})$? (Hint $\sum_{k=0}^{n-1} 2^k = 2^n - 1$). (1.5 pt)