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(1) Consider a 2-period binomial model with $S_0 = 10$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = \mathbb{P}(T)$.

(a) Consider a European option with payoff $V_2 = \max\left((S_0, S_1, S_2) - 10\right)^+$. Determine the price V_n at time $n = 0, 1, 2$. (0.75 pt)

(b) Consider the utility function $U(x) = \ln(2x + 1)$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $\mathbb{E}(U(X))$ subject to the condition that $\widetilde{\mathbb{E}}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

$$X = X_2 = \frac{1}{Z}\left[(1.1)^2 X_0 + \frac{1}{2}\right] - \frac{1}{2}.$$

(1 pt)

(c) Consider part (b) and assume $X_0 = 100$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$. (1.25 pt)

(d) Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 12 - \max\Big(S_0, \cdots, S_n\Big)$, $n = 0, 1, 2$. Determine the price V_n at time $n = 0, 1, 2$ of the American option. Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$. (1 pt)

Solution (a): We first calculate the risk-neutral probability measure $\widetilde{\mathbb{P}}$, we have $\widetilde{\mathbb{P}}(H) = \widetilde{p} = 3/4$ and $\widetilde{\mathbb{P}}(T) = \widetilde{q} = 1/4$. We start with the value of V_2, we have $V_2(HH) = 4.4, V_2(HT) = 2, V_2(TH) = 0, V_2(TT) = 0$. Then

$$V_1(H) = \frac{1}{1.1}[\frac{3}{4}(4.4) + \frac{1}{4}(2)] = 3.45,$$

and

$$V_1(T) = \frac{1}{1.1}[\frac{3}{4}(0) + \frac{1}{4}(0)] = 0,$$

leading to

$$V_0 = \frac{1}{1.1}[\frac{3}{4}(3.45) + \frac{1}{4}(0)] = 2.36.$$

Solution (b): Notice that the function $U(x) = \ln(2x + 1)$, $x > 0$ is strict concave with $U'(x) = \frac{2}{2x+1}$. We apply Theorem 3.3.6, we find that the inverse I of U' is given by $I(x) = \frac{1}{x} - \frac{1}{2}$. Thus, the optimal solution is given by

$$X_2 = X = I\left(\frac{\lambda Z}{(1.1)^2}\right) = \frac{(1.1)^2}{\lambda Z} - \frac{1}{2},$$

and satisfies the constraint

$$X_0 = \mathbb{E}\left(\frac{XZ}{(1.1)^2}\right) = \frac{1}{\lambda} - \frac{\mathbb{E}(Z)}{2(1.1)^2} = \frac{1}{\lambda} - \frac{1}{2(1.1)^2},$$

where the last equality follows from the fact that $\mathbb{E}(Z) = 1$. Hence,

$$\frac{1}{\lambda} = X_0 + \frac{1}{2(1.1)^2},$$

and

$$X = X_2 = \frac{1}{Z}\left[(1.1)^2 X_0 + \frac{1}{2}\right] - \frac{1}{2}.$$

Solution (c): To find the optimal portfolio and corresponding wealth processes, we first determine explicitly the the random variable $X = X_2$, and then we apply Theorem 1.2.2 with $X_0 = 100$. We begin by find the Radon Nikodym derivative Z. We have

$$Z(HH) = \frac{9}{4},\ Z(HT) = Z(TH) = \frac{3}{4},\ Z(TT) = \frac{1}{4}.$$

From part (b) with $X_0 = 100$, we have

$$X = X_2 = \frac{121.5}{Z} - \frac{1}{2}.$$

This leads to

$$X_2(HH) = 53.5,\ X_2(HT) = X_2(TH) = 161.5,\ X_2(TT) = 485.5.$$

Hence,

$$X_1(H) = \frac{1}{1.1}[\frac{3}{4}(53.5) + \frac{1}{4}(161.5)] = 73.182,$$
$$X_1(T) = \frac{1}{1.1}[\frac{3}{4}(161.5) + \frac{1}{4}(485.5)] = 220.455.$$

Notice that

$$X_0 = \frac{1}{1.1}[\frac{3}{4}(73.182) + \frac{1}{4}(220.455)] = 100$$

as required. The optimal portfolio is given by

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{72.182 - 220.455}{12 - 8} = -37.07,$$
$$\Delta_(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{53.5 - 161.5}{14.4 - 9.6} = -22.5,$$
$$\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2T(T)} = \frac{161.5 - 485.5}{9.6 - 6.4} = -101.25.$$

Solution (d): The intrinsic value process is given by

$$G_0 = 0, G_1(H) = 0, G_1(T) = 2,$$
$$G_2(HH) = 0, G_2(HT) = 0.4, G_2(TH) = 2, G_2(TT) = 3.6.$$

The payoff at time 2 is given by

$$V_2(HH) = 0,\ V_2(HT) = 0.4,\ V_2(TH) = 2,\ V_2(TT) = 3.6.$$

Applying the American algorithm, we get

$$V_1(H) = \max\left(0, \frac{1}{1.1}[(\frac{3}{4})(0) + (\frac{1}{4})(0.4)]\right) = \max(0, 0.091) = 0.091,$$
$$V_1(T) = \max\left(2, \frac{1}{1.1}[(\frac{3}{4})(2) + (\frac{1}{4})(3.6)]\right) = \max(2, 2, 182) = 2.182,$$
$$V_0 = \max\left(0, \frac{1}{1.1}[(\frac{3}{4})(0.091) + (\frac{1}{4})(2.182)]\right) = \max(0, 0.588) = 0.588.$$

The optimal exercise time is given by

$$\tau^*(HH) = \infty,\ \tau^*(HT) = \tau^*(TH) = \tau^*(TT) = 2.$$

(2) Consider an N-period binomial model with real probability measure $\mathbb{P}$ satisfisfying $\mathbb{P}(H) = p$ and $q = 1 - p = \mathbb{P}(T)$. For $n = 1, \cdots, N$ define

$$Y_n = \begin{cases} 2, & \text{if } \omega_n = H, \\ -3, & \text{if } \omega_n = T. \end{cases}$$

Set $M_0 = 0$ and let $M_n = \sum_{i=1}^{n} Y_i$, $n = 1, \cdots, N$.

(a) Prove that

$$\mathbb{P}(M_n = k) = \begin{cases} \binom{n}{\frac{3n+k}{5}} p^{(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n+k \equiv 0 \bmod 5, \\ 0, & \text{otherwise .} \end{cases}$$

(1 pt)

(b) Define $U_n = M_n Y_n$ for $n = 1, \cdots, N$. Show that for any function $f : \mathbb{R}^2 \to \mathbb{R}$, and any $n = 0, 1, \cdots, N$,

$$\mathbb{E}_n\Big(f(U_{n+1}, Y_{n+1})\Big) = pf\Big(\frac{2U_n}{Y_n} + 4, 2\Big) + qf\Big(\frac{-3U_n}{Y_n} + 9, -3\Big).$$

Conclude that $(U_1, Y_1), \cdots, (U_N, Y_N)$ is a Markov process under $\mathbb{P}$. (1.25 pt)

(c) For which values of p is the process $\{M_n : n = 0, 1, \cdots, N\}$ a (i) martingale, (ii) submartingale, (iii) supermartingale? (1 pt)

(d) Suppose $p = \mathbb{P}(H) = 3/5$ and $q = 1 - p = \mathbb{P}(T) = 2/5$. Define the stopping time τ by

$$\tau = \inf\{n \geq 0 : M_n = 3\}.$$

Determine the value of $\mathbb{E}(M_{n\wedge\tau})$ for $n = 0, 1, \cdots, N$. (0.5 pt)

Solution (a): Let r_n be the number of right steps and ℓ_n the number of left steps at time n. Then, $r_n + \ell_n = n$ and $2r_n - 3\ell_n = k$. Solving these two equations simultaneously, we get $r_n = \dfrac{3n+k}{5}$ and $\ell_n = \dfrac{2n-k}{5}$. Since ℓ_n and r_n must be integers, this makes sense only when $3n + k \equiv 0 \bmod 5$. Assuming that $3n + k \equiv 0 \bmod 5$, we see that the number of such paths is $\binom{n}{\frac{3n+k}{5}}$, and the probability of each such path is $p^{(3n+k)/5}q^{(2n-k)/5}$. Therefore,

$$\mathbb{P}(M_n = k) = \begin{cases} \binom{n}{\frac{3n+k}{5}} p^{(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n+k \equiv 0 \bmod 5, \\ 0, & \text{otherwise .} \end{cases}$$

Solution (b): First note that the process (U_n, Y_n) is adapted and

$$U_{n+1} = M_{n+1}Y_{n+1} = (M_n + Y_{n+1})Y_{n+1} = M_n Y_n \frac{Y_{n+1}}{Y_n} + Y_{n+1}^2 = U_n \frac{Y_{n+1}}{Y_n} + Y_{n+1}^2$$

with U_n, Y_n depending on the first n coin tosses and Y_{n+1} is independent of the first n tosses. Hence, for any function $f : \mathbb{R}^2 \to \mathbb{R}$, we have by the Independence Lemma

$$\mathbb{E}_n\Big(f(U_{n+1}, Y_{n+1})\Big) = \mathbb{E}_n\Big(f(U_n \frac{Y_{n+1}}{Y_n} + Y_{n+1}^2, Y_{n+1}\Big) = g(U_n, Y_n),$$

with

$$g(u, y) = \mathbb{E}\Big(f(\frac{uY_{n+1}}{y} + Y_{n+1}^2, Y_{n+1}\Big) = pf(\frac{2u}{y} + 4, 2) + qf(\frac{-3u}{y} + 9, -3).$$

Thus,

$$\mathbb{E}_n\Big(f(U_{n+1}, Y_{n+1})\Big) = pf\Big(\frac{2U_n}{Y_n} + 4, 2\Big) + qf\Big(\frac{-3U_n}{Y_n} + 9, -3\Big).$$

Since $\mathbb{E}_n\Big(f(U_{n+1}, Y_{n+1})\Big)$ depends only on (U_n, Y_n), we see that $(U_1, Y_1), \cdots, (U_N, Y_N)$ is a Markov process under $\mathbb{P}$.

Solution (c): First note that $(M_n : n = 0, 1, \cdots)$ is an adapted process. An easy calculation shows that

$$\mathbb{E}_n(M_{n+1}) = p(M_n + 2) + (1 - p)(M_n + 3) = M_n + 5p - 3.$$

(i) For $p = \frac{3}{5}$ we have $\mathbb{E}_n\Big(M_{n+1}\Big) = M_n$ and therefore $\{M_n : n = 0, 1, \cdots, N\}$ is a martingale.

(ii) For $p \geq \frac{3}{5}$ we have $\mathbb{E}_n\Big(M_{n+1}\Big) \geq M_n$ and therefore $\{M_n : n = 0, 1, \cdots, N\}$ is a submartingale.

(iii) For $p \leq \frac{3}{5}$ we have $\mathbb{E}_n\Big(M_{n+1}\Big) \leq M_n$ and therefore $\{M_n : n = 0, 1, \cdots, N\}$ is a supermartingale.

Solution (d): From part (c, i), with $p = \frac{3}{5}$ the process $\{M_n : n = 0, 1, \cdots, N\}$ is a martingale By the Optional Sampling: Part I, the stopped process $(M_{n\wedge\tau} : n = 0, 1, \cdots)$ is a martingale, hence (one can also apply the Optional Sampling: Part II directly)

$$\mathbb{E}[M_{n\wedge\tau}] = \mathbb{E}[M_{0\wedge\tau}] = \mathbb{E}[M_0] = 0.$$

(3) Consider the (infinite) binomial model with up factor $u = \sqrt{2}$, down factor $d = \dfrac{1}{\sqrt{2}}$ and interest rate $r = \dfrac{3\sqrt{2}}{4} - 1$. Suppose the real probability $\mathbb{P}$ is given by $\mathbb{P}(H) = p = 2/3$ and $\mathbb{P}(T) = q = 1/3$. Define the process $(M_n : n = 0, 1, \cdots)$ by $M_0 = 0$ and $M_n = \sum_{i=1}^{n} X_i$ where

$$X_i = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

Consider a perpetual American put option with $S_0 = 4$ and strike price $K = 8$

(a) Show that the price process $(S_n : n = 0, 1, \cdots)$ is given by $S_n = 2^{2+\frac{1}{2}M_n}$, and the risk-neutral probability $\widetilde{\mathbb{P}}$ is given by $\widetilde{\mathbb{P}}(H) = \widetilde{p} = 1/2 = \widetilde{q} = \widetilde{\mathbb{P}}(T)$. (1 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 2 euros. What is then the price at time 0 of such an option? (0.75 pt)

(c) Determine under $\mathbb{P}$, the probability that the price reaches $4\sqrt{2}$ euros for the first time at time $n = 5$? (0.5 pt)

Solution (a): In the binomial model the price at time n is given by

$$S_n(\omega_1 \cdots \omega_n) = S_0\, u^{\#H(\omega_1\cdots\omega_n)}\, d^{\#T(\omega_1\cdots\omega_n)}.$$

Now,

$$u^{\#H(\omega_1\cdots\omega_n)} = 2^{\dfrac{1}{2}\sum\limits_{1\leq i\leq n: X_i = 1} X_i(\omega_i)},$$

and

$$d^{\#T(\omega_1\cdots\omega_n)} = 2^{\dfrac{1}{2}\sum\limits_{1\leq i\leq n: X_i = -1} X_i(\omega_i)}.$$

Thus,

$$S_n(\omega_1 \cdots \omega_n) = S_0\, 2^{\frac{1}{2}M_n(\omega_1\cdots\omega_n)} = 2^{2+\frac{1}{2}M_n(\omega_1\cdots\omega_n)}.$$

This shows that $S_n = 2^{2+\frac{1}{2}M_n}$. We now calculate the risk-neutral probability, we have

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{\frac{3\sqrt{2}}{4} - \frac{1}{\sqrt{2}}}{\sqrt{2} - \frac{1}{\sqrt{2}}} = \frac{1}{2}.$$

Note that under the risk-neutral probability $\widetilde{\mathbb{P}}$, the process $(M_n : n = 0, 1, \cdots)$ is a symmetric random walk.

Solution (b): Note that $S_n = 2 = 2^1$ for the first time if and only if $M_n = -2$ for the first time. Thus, the buyer is using the exercise policy τ_{-2}. Hence, the price at time 0 should be

$$V_0 = V^{\tau_{-2}} = \widetilde{\mathbb{E}}\left(\left(\frac{1}{1+r}\right)^{\tau_{-2}}(8 - S_{\tau_{-2}})\right) = 6\,\widetilde{\mathbb{E}}\left(\left(\frac{4}{3\sqrt{2}}\right)^{\tau_{-2}}\right) = 6\,\widetilde{\mathbb{E}}\left(\left(\frac{2\sqrt{2}}{3}\right)^{\tau_{-2}}\right).$$

To calculate the expectation, we use Theorem 5.2.3 with $\alpha = \frac{2\sqrt{2}}{3}$, and this leads to

$$V_0 = V^{\tau_{-2}} = 6\,\widetilde{\mathbb{E}}\left(\left(\frac{2\sqrt{2}}{3}\right)^{\tau_{-2}}\right) = 6\left(\frac{1}{\sqrt{2}}\right)^2 = 3.$$

Solution (c): The probability that the price reaches $4\sqrt{2}$ for the first time at time 5 is equal to the probability that the random walk reaches level 1 for the first time at time 5. Equivalently, we are looking for $\mathbb{P}(\tau_1 = 5)$. By Theorem 5.2.5,

$$\mathbb{P}(\{\tau_1 = 5\}) = \frac{4!}{3!\,2!}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{16}{243}.$$