(1) Consider a 2-period binomial model with $S_0 = 10$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $P$ satisfies $P(H) = p = \frac{1}{2} = P(T)$.

(a) Consider a European option with payoff $V_2 = \max((S_0, S_1, S_2) - 10)^+$. Determine the price $V_n$ at time $n = 0, 1, 2$. (0.75 pt)

(b) Consider the utility function $U(x) = \ln(2x + 1)$ ($x > 0$). Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\tilde{E}\left(\left(\frac{X}{1 + r}\right)^2\right) = X_0$ is given by

\[
X = X_2 = \frac{1}{Z} \left(\frac{X}{1 + r}\right)^2 - \frac{1}{2}.
\]

(1 pt)

(c) Consider part (b) and assume $X_0 = 100$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$. (1.25 pt)

(d) Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 12 - \max(S_0, \cdots, S_n)$, $n = 0, 1, 2$. Determine the price $V_n$ at time $n = 0, 1, 2$ of the American option. Find the optimal exercise time $\tau^*(\omega_1 \omega_2)$ for all $\omega_1 \omega_2$. (1 pt)

Solution (a): We first calculate the risk-neutral probability measure $\tilde{P}$, we have $\tilde{P}(H) = \tilde{p} = \frac{3}{4}$ and $\tilde{P}(T) = \tilde{q} = 1/4$. We start with the value of $V_2$, we have $V_2(HH) = 4.4, V_2(HT) = 2, V_2(TH) = 0, V_2(TT) = 0$. Then

\[
V_1(H) = \frac{1}{1.1} \left[\frac{3}{4}(4.4) + \frac{1}{4}(2)\right] = 3.45,
\]

and

\[
V_1(T) = \frac{1}{1.1} \left[\frac{3}{4}(0) + \frac{1}{4}(0)\right] = 0,
\]

leading to

\[
V_0 = \frac{1}{1.1} \left[\frac{3}{4}(3.45) + \frac{1}{4}(0)\right] = 2.36.
\]

Solution (b): Notice that the function $U(x) = \ln(2x + 1)$, $x > 0$ is strict concave with $U'(x) = \frac{2}{2x + 1}$. We apply Theorem 3.3.6, we find that the inverse $I$ of $U'$ is given by $I(x) = \frac{1}{x} - \frac{1}{2}$. Thus, the optimal solution is given by

\[
X_2 = X = I\left(\frac{X}{(1.1)^2}\right) = \frac{1}{X} - \frac{1}{2},
\]

and satisfies the constraint

\[
X_0 = E\left(\frac{XZ}{(1.1)^2}\right) = \frac{1}{X} - \frac{E(Z)}{2(1.1)^2} = \frac{1}{X} - \frac{1}{2(1.1)^2},
\]

where the last equality follows from the fact that $E(Z) = 1$. Hence,

\[
\frac{1}{X} = X_0 + \frac{1}{2(1.1)^2},
\]

and

\[
X = X_2 = \frac{1}{Z} \left(\frac{(1.1)^2 X_0 + 1}{2}\right) - \frac{1}{2}.
\]
The optimal exercise time is given by \( H \). Hence, as required. The optimal portfolio is given by \( X \). Notice that \( X \). From part (b) with \( X \), we find the Radon Nikodym derivative \( Z \). We have
\[
Z(HH) = \frac{9}{4}, \ Z(HT) = Z(TH) = \frac{3}{4}, \ Z(TT) = \frac{1}{4}.
\]
From part (b) with \( X \), we have
\[
X = X_2 = \frac{121.5}{Z} - \frac{1}{2}.
\]
This leads to
\[
X_2(HH) = 53.5, \ X_2(HT) = X_2(TH) = 161.5, \ X_2(TT) = 485.5.
\]
Hence,
\[
X_1(H) = \frac{1}{1.1} \left[ \frac{3}{4}(53.5) + \frac{1}{4}(161.5) \right] = 73.182,
\]
\[
X_1(T) = \frac{1}{1.1} \left[ \frac{3}{4}(161.5) + \frac{1}{4}(485.5) \right] = 220.455.
\]
Notice that
\[
X_0 = \frac{1}{1.1} \left[ \frac{3}{4}(73.182) + \frac{1}{4}(220.455) \right] = 100
\]
as required. The optimal portfolio is given by
\[
\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{72.182 - 220.455}{12 - 8} = -37.07,
\]
\[
\Delta_1(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{53.5 - 161.5}{14.4 - 9.6} = -22.5,
\]
\[
\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} = \frac{161.5 - 485.5}{9.6 - 6.4} = -101.25.
\]
Solution (d): The intrinsic value process is given by
\[
G_0 = 0, \ G_1(H) = 0, \ G_1(T) = 2,
\]
\[
G_2(HH) = 0, \ G_2(HT) = 0.4, \ G_2(TH) = 2, \ G_2(TT) = 3.6.
\]
The payoff at time 2 is given by
\[
V_2(HH) = 0, \ V_2(HT) = 0.4, \ V_2(TH) = 2, \ V_2(TT) = 3.6.
\]
Applying the American algorithm, we get
\[
V_1(H) = \max \left( 0, \ \frac{1}{1.1} \left[ \frac{3}{4}(0) + \frac{1}{4}(0.4) \right] \right) = \max(0, 0.091) = 0.091,
\]
\[
V_1(T) = \max \left( 2, \ \frac{1}{1.1} \left[ \frac{3}{4}(2) + \frac{1}{4}(3.6) \right] \right) = \max(2, 2.182) = 2.182,
\]
\[
V_0 = \max \left( 0, \ \frac{1}{1.1} \left[ \frac{3}{4}(0.091) + \frac{1}{4}(2.182) \right] \right) = \max(0, 0.588) = 0.588.
\]
The optimal exercise time is given by
\[
\tau^*(HH) = \infty, \ \tau^*(HT) = \tau^*(TH) = \tau^*(TT) = 2.
\]
(2) Consider an \( N \)-period binomial model with real probability measure \( \mathbb{P} \) satisfying \( \mathbb{P}(H) = p \) and \( q = 1 - p = \mathbb{P}(T) \). For \( n = 1, \ldots, N \) define
\[
Y_n = \begin{cases} 
2, & \text{if } \omega_n = H, \\
-3, & \text{if } \omega_n = T.
\end{cases}
\]
Set \( M_0 = 0 \) and let \( M_n = \sum_{i=1}^{n} Y_i, \ n = 1, \ldots, N. \)
(a) Prove that
\[ P(M_n = k) = \begin{cases} \left(\frac{n}{3n+k}\right)^{p(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n + k \equiv 0 \mod 5, \\ 0, & \text{otherwise}. \end{cases} \]

(1 pt)

(b) Define \( U_n = M_n Y_n \) for \( n = 1, \cdots, N \). Show that for any function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), and any \( n = 0, 1, \cdots, N \),
\[ E_n(f(U_{n+1}, Y_{n+1})) = p f\left(\frac{2U_n}{Y_n} + 4, 2\right) + q f\left(-\frac{3U_n}{Y_n} + 9, -3\right). \]

Conclude that \( (U_1, Y_1), \cdots, (U_N, Y_N) \) is a Markov process under \( P \). (1.25 pt)

(c) For which values of \( p \) is the process \( \{M_n : n = 0, 1, \cdots, N\} \) a (i) martingale, (ii) submartingale, (iii) supermartingale? (1 pt)

(d) Suppose \( p = \mathbb{P}(H) = 3/5 \) and \( q = 1 - p = \mathbb{P}(T) = 2/5 \). Define the stopping time \( \tau \) by
\[ \tau = \inf\{n \geq 0 : M_n = 3\}. \]

Determine the value of \( \mathbb{E}(M_{n \land \tau}) \) for \( n = 0, 1, \cdots, N \). (0.5 pt)

**Solution (a):** Let \( r_n \) be the number of right steps and \( \ell_n \) the number of left steps at time \( n \). Then, \( r_n + \ell_n = n \) and \( 2r_n - 3\ell_n = k \). Solving these two equations simultaneously, we get \( r_n = \frac{3n + k}{5} \) and \( \ell_n = \frac{2n - k}{5} \). Since \( \ell_n \) and \( r_n \) must be integers, this makes sense only when \( 3n + k \equiv 0 \mod 5 \). Assuming that \( 3n + k \equiv 0 \mod 5 \), we see that the number of such paths is \( \left(\frac{n}{3n+k}\right)^{p(3n+k)/5} q^{(2n-k)/5} \). Therefore,
\[ P(M_n = k) = \begin{cases} \left(\frac{n}{3n+k}\right)^{p(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n + k \equiv 0 \mod 5, \\ 0, & \text{otherwise}. \end{cases} \]

**Solution (b):** First note that the process \( (U_n, Y_n) \) is adapted and
\[ U_{n+1} = M_{n+1} Y_{n+1} = (M_n + Y_{n+1}) Y_n = M_n Y_n Y_{n+1} + Y_{n+1}^2 = U_n Y_{n+1}^2 + Y_{n+1}^2 \]
with \( U_n, Y_n \) depending on the first \( n \) coin tosses and \( Y_{n+1} \) is independent of the first \( n \) tosses. Hence, for any function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), we have by the Independence Lemma
\[ E_n\left(f(U_{n+1}, Y_{n+1})\right) = E_n\left(f\left(U_n Y_{n+1} + Y_{n+1}^2, Y_{n+1}\right)\right) = g(U_n, Y_n), \]
with
\[ g(u, y) = E\left(f\left(\frac{uY_{n+1}}{y} + Y_{n+1}^2, Y_{n+1}\right)\right) = p f\left(\frac{2u}{y} + 4, 2\right) + q f\left(-\frac{3u}{y} + 9, -3\right). \]

Thus,
\[ E_n\left(f(U_{n+1}, Y_{n+1})\right) = p f\left(\frac{2U_n}{Y_n} + 4, 2\right) + q f\left(-\frac{3U_n}{Y_n} + 9, -3\right). \]

Since \( E_n\left(f(U_{n+1}, Y_{n+1})\right) \) depends only on \( (U_n, Y_n) \), we see that \( (U_1, Y_1), \cdots, (U_N, Y_N) \) is a Markov process under \( P \).

**Solution (c):** First note that \( \{M_n : n = 0, 1, \cdots\} \) is an adapted process. An easy calculation shows that
\[ E_n(M_{n+1}) = p(M_n + 2) + (1 - p)(M_n + 3) = M_n + 5p - 3. \]
(i) For \( p = \frac{3}{5} \) we have \( \mathbb{E}_n(M_{n+1}) = M_n \) and therefore \( \{M_n : n = 0, 1, \ldots, N\} \) is a martingale.

(ii) For \( p \geq \frac{2}{3} \) we have \( \mathbb{E}_n(M_{n+1}) \geq M_n \) and therefore \( \{M_n : n = 0, 1, \ldots, N\} \) is a submartingale.

(iii) For \( p \leq \frac{3}{5} \) we have \( \mathbb{E}_n(M_{n+1}) \leq M_n \) and therefore \( \{M_n : n = 0, 1, \ldots, N\} \) is a supermartingale.

**Solution (d):** From part (c, i), with \( p = \frac{3}{5} \) the process \( \{M_n : n = 0, 1, \ldots, N\} \) is a martingale. By the Optional Sampling: Part I, the stopped process \( (M_{n\wedge \tau} : n = 0, 1, \ldots) \) is a martingale, hence (one can also apply the Optional Sampling: Part II directly)

\[
\mathbb{E}[M_{n\wedge \tau}] = \mathbb{E}[M_0\wedge \tau] = \mathbb{E}[M_0] = 0.
\]

(3) Consider the (infinite) binomial model with up factor \( u = \sqrt{2} \), down factor \( d = \frac{1}{\sqrt{2}} \) and interest rate \( r = \frac{3\sqrt{2}}{4} - 1 \). Suppose the real probability \( \mathbb{P} \) is given by \( \mathbb{P}(H) = p = \frac{2}{3} \) and \( \mathbb{P}(T) = q = \frac{1}{3} \).

Define the process \( \{M_n : n = 0, 1, \ldots\} \) by \( M_0 = 0 \) and \( M_n = \sum_{i=1}^{n} X_i \) where

\[
X_i = \begin{cases} 
1, & \text{if } \omega_i = H, \\
-1, & \text{if } \omega_i = T,
\end{cases}
\]

Consider a perpetual American put option with \( S_0 = 4 \) and strike price \( K = 8 \)

(a) Show that the price process \( \{S_n : n = 0, 1, \ldots\} \) is given by \( S_n = 2^{2+\frac{1}{2}M_n} \), and the risk-neutral probability \( \mathbb{P} \) is given by \( \mathbb{P}(H) = \tilde{p} = 1/2 = \tilde{q} = \mathbb{P}(T) \). (1 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 2 euros. What is then the price at time 0 of such an option? (0.75 pt)

(c) Determine under \( \mathbb{P} \), the probability that the price reaches \( 4\sqrt{2} \) euros for the first time at time \( n = 5 \? \) (0.5 pt)

**Solution (a):** In the binomial model the price at time \( n \) is given by

\[
S_n(\omega_1 \cdots \omega_n) = S_0 u^{\#H(\omega_1 \cdots \omega_n)} d^{\#T(\omega_1 \cdots \omega_n)}.
\]

Now,

\[
u^{\#H(\omega_1 \cdots \omega_n)} = 2^{\frac{1}{2} \sum_{1 \leq i \leq n : X_i = 1} X_i(\omega_i)},
\]

and

\[
d^{\#T(\omega_1 \cdots \omega_n)} = 2^{\frac{1}{2} \sum_{1 \leq i \leq n : X_i = -1} X_i(\omega_i)}.
\]

Thus,

\[
S_n(\omega_1 \cdots \omega_n) = S_0 2^{\frac{1}{2}M_n(\omega_1 \cdots \omega_n)} = 2^{2+\frac{1}{2}M_n(\omega_1 \cdots \omega_n)}.
\]

This shows that \( S_n = 2^{2+\frac{1}{2}M_n} \). We now calculate the risk-neutral probability, we have

\[
\tilde{p} = \frac{1 + r - d}{u - d} = \frac{3\sqrt{2} - \frac{1}{\sqrt{2}}}{\sqrt{2} - \frac{1}{\sqrt{2}}} = \frac{1}{2}.
\]

Note that under the risk-neutral probability \( \mathbb{P} \), the process \( \{M_n : n = 0, 1, \ldots\} \) is a symmetric random walk.
Solution (b): Note that $S_n = 2 = 2^1$ for the first time if and only if $M_n = -2$ for the first time. Thus, the buyer is using the exercise policy $\tau_{-2}$. Hence, the price at time 0 should be

$$V_0 = V^{\tau_{-2}} = \mathbb{E}\left(\left(\frac{1}{1+r}\right)^{\tau_{-2}}(8-S_{\tau_{-2}})\right) = 6\mathbb{E}\left(\left(\frac{4}{3\sqrt{2}}\right)^{\tau_{-2}}\right) = 6\mathbb{E}\left(\left(\frac{2\sqrt{2}}{3}\right)^{\tau_{-2}}\right).$$

To calculate the expectation, we use Theorem 5.2.3 with $\alpha = \frac{2\sqrt{2}}{3}$, and this leads to

$$V_0 = V^{\tau_{-2}} = 6\mathbb{E}\left(\left(\frac{2\sqrt{2}}{3}\right)^{\tau_{-2}}\right) = 6\left(\frac{1}{\sqrt{2}}\right)^2 = 3.$$

Solution (c): The probability that the price reaches $4\sqrt{2}$ for the first time at time 5 is equal to the probability that the random walk reaches level 1 for the first time at time 5. Equivalently, we are looking for $\mathbb{P}(\tau_1 = 5)$. By Theorem 5.2.5,

$$\mathbb{P}(\{\tau_1 = 5\}) = \frac{4!}{3!2!}\cdot\frac{2}{3}\cdot\left(\frac{1}{3}\right)^2 = \frac{16}{243}.$$