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Solutions Retake Exam: Inleiding Financiele Wiskunde 2018-2019

(1) Let $(W(t) : t \geq 0)$ be a Brownian motion, define a process $(X(t) : t \geq 0)$ by $X(t) = \frac{1}{\sqrt{2}} W(2t)$.

(a) Prove that $(X(t) : t \geq 0)$ is a Brownian motion. (1 pt)

(b) Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $(Y(t) : t \geq 0)$ a martingale with respect to the filtration $(\mathcal{F}(t) : t \geq 0)$ with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$. (1 pt)

(c) Consider the process $\{Z(t) : t \geq 0\}$ defined by $Z(t) = \int_0^t e^u \, dX(u)$. Determine the distribution of $Z(t)$ and calculate $\mathbb{P}(Z(t) \leq 1)$. (1 pt)

Proof (a): We check that the process $(X(t) : t \geq 0)$ satisfies all the properties of a Brownian motion. We have

(i) $X(0) = \frac{1}{\sqrt{2}} W(0) = 0$.

(ii) Since $(W(t) : t \geq 0)$ has continuous paths and the function $t \to 2tt$ is continuous and so is multiplication by $1/\sqrt{2}$, we see that the process $(X(t) : t \geq 0)$ also has continuous paths.

(iii) If $0 \leq u < v < s < t$, then clearly $0 \leq 2u < 2v < 2s < 2t$. Hence $W(2v) - W(2u)$ and $W(2t) - W(2s)$ are independent implying that $X(v) - X(u)$ and $X(t) - X(s)$ are independent. Therefore the process $(X(t) : t \geq 0)$ has independent increment.

(iv) if $s < t$, then $W(2t) - W(s)$ is normally distributed with mean zero and variance $2t - 2s = 2(t - s)$. hence $X(t) - X(s)$ is also normally distributed with mean

$$\mathbb{E}(X(t) - X(s)) = \frac{1}{\sqrt{2}} \mathbb{E}(W(2t) - W(2s)) = 0,$$

and variance

$$\text{Var}(X(t) - X(s)) = \frac{1}{2} \text{Var}(W(2t) - W(2s)) = (t - s).$$

Therefore, $(X(t) : t \geq 0)$ is a Brownian motion.

Proof (b): The underlying filtration is given by $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$. Now let $s < t$, and note that $Y(t) - Y(s)$ is independent of $\mathcal{F}(s)$ while $Y(s)$ is $\mathcal{F}(s)$-measurable. Hence,

$$\begin{aligned}
\mathbb{E}[Y(t)|\mathcal{F}(s)] &= \mathbb{E}[(Y(t) - Y(s)) + Y(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[Y(t) - Y(s))] + Y(s) \\
&= \mathbb{E}[X^2(t) - X^2(s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= \mathbb{E}[\frac{1}{2} W^2(2t) - \frac{1}{2} W^2(2s)] - 2\sqrt{c}(t - s) + Y(s) \\
&= \frac{1}{2} 2t - \frac{1}{2} 2s - 2\sqrt{c}(t - s) + Y(s) \\
&= (1 - 2\sqrt{c})(t - s) + Y(s).
\end{aligned}$$

Hence, for the process $(Y(t) : t \geq 0)$ to be a martingale, we must have $c = 1/4$.

Proof (c): Note that $Z(t)$ is an Itô-integral of a deterministic process, hence it is normally distributed with mean 0 and variance

$$\mathrm{Var}\Big(\int_0^t e^u\,dW(u)\Big) = \int_0^t e^{2u}\,du = \frac{1}{2}(e^{2t}-1).$$

Thus,

$$\mathbb{P}(Z(t)\le 1) = \mathbb{P}\Big(\frac{Z(t)}{\sqrt{\frac{1}{2}(e^{2t}-1)}} \le \frac{1}{\sqrt{\frac{1}{2}(e^{2t}-1)}}\Big) = N\Big(\frac{1}{\sqrt{\frac{1}{2}(e^{2t}-1)}}\Big),$$

where $N(x)$ is the standard normal distribution function.

(2) Let $(W(t):t\ge 0)$ be a Brownian motion and let $\{\mathcal{F}(t):t\ge 0\}$ be its natural filtration. Consider the Itô process $\{X(t):t\ge 0\}$ with

$$X(t) = X(0) + \int_0^t \alpha X(u)\,du + \int_0^t 2X(u)\,dW(u)$$

with α and $X(0)$ some constants.

(a) Show that $\mathbb{E}[X(t)] = X(0)e^{\alpha t}$. (Hint: you are allowed to interchange the integral and the expectation) (1 pt)

(b) Show that the process $\{X^2(t):t\ge 0\}$ is an Itô process. (1 pt)

(c) For which values of α is the process $\{X^2(t):t\ge 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t):t\ge 0\}$? (1 pt)

Proof (a): Since an Itô integral has zero expectation, we see that

$$\mathbb{E}[X(t)] = X(0) + \mathbb{E}\Big[\int_0^t \alpha X(u)\,du\Big] = X(0) + \int_0^t \alpha\mathbb{E}[X(u)]\,du.$$

Let $m(t) = \mathbb{E}[X(t)$, then the above equation reads $m(t) = X(0) + \int_0^t \alpha m(u)\,du$. Equivalently, $\frac{dm(t)}{dt} = \alpha m(t)$ which has solution $\mathbb{E}[X(t)] = m(t) = X(0)e^{\alpha t}$.

Proof (b): We apply Itô Doeblin formula to the function $f(x)=x^2$. We have $f_x(x)=2x$, $f_{xx}(x)=2$ and $dX(t)dX(t) = 4X^2(t)\,dt$. Hence,

$$\begin{aligned} dX^2(t) &= df(X(t)) = 2X(t)dX(t) + 4X^2(t)\,dt \\ &= 2X(t)[\alpha X(t)\,dt + 2X(t)\,dW(t)] + 4X^2(t)\,dt \\ &= (2\alpha+4)X^2(t)\,dt + 4X^2(t)\,dW(t). \end{aligned}$$

Equivalently,

$$X^2(t) = X^2(0) + \int_0^t (2\alpha+4)X^2(u),du + \int_0^t 4X^2(u)\,dW(u).$$

Therefore, $\{X^2(t):t\ge 0\}$ is an Itô process.

Proof (c): Since Itô integrals are martingales, from part (b) we see that $\{X^2(t):t\ge 0\}$ is a martingale if $2\alpha+4=0$, which implies that $\alpha=-2$.

(3) Let $\{W(t):0\le t\le T\}$ be a Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let $\{\mathcal{F}(t):0\le t\le T\}$ be its natural filtration, and assume $\mathcal{F}=\mathcal{F}(T)$. Consider a stock with price process $\{S(t):0\le t\le T\}$ with $S(t)=t^3+3W(t)$.

(a) Construct a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\widetilde{\mathbb{P}}(A)=0$ if and only if $\mathbb{P}(A)=0$, $A\in\mathcal{F}$) such that the price process $\{S(t):0\le t\le T\}$ is a martingale under $\widetilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t):0\le t\le T\}$. (1 pt)

(b) Consider a European call option on this stock with expiration date T and strike price K. Find an expression for $C(0)=\widetilde{\mathbb{E}}\Big[(S(T)-K)^+\Big]$, the price of this option at time 0. (1 pt)

Proof (a) : Define $\theta(t) = t^2$, then $S(t)/3 = \int_0^t \theta(u)\,du + W(t)$. Consider the random variable Z defined by

$$Z = \exp\Big(- \int_0^T \theta(u)\,dW(u) - \frac{1}{2}\int_0^T \theta^2(u)\,du \Big) = \exp\Big(- \int_0^T t^2\,dW(u) - \frac{1}{10}t^5 \Big).$$

Note that $\int_0^t \theta(u)\,dW(u)$, $\int_0^t \theta^2(u)\,du$ and θ are continuous functions on the compact interval $[0,T]$, hence they are all bounded. This implies that $\mathbb{E}\Big[\int_0^T \theta^2(u) Z^2(u)\,du\Big] < \infty$. Define the measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}$ by $\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P}$. By Girsanov's Theorem, the process $\{\frac{S(t)}{3} : 0 \le t \le T\}$ is a Brownian motion under $\widetilde{\mathbb{P}}$ and hence is a martingale under $\widetilde{\mathbb{P}}$. Therefore, $\{S(t) : 0 \le t \le T\}$ is a martingale under $\widetilde{\mathbb{P}}$.

Proof (b) : By part (a), we see that under the measure $\widetilde{\mathbb{P}}$, the random variable $\frac{S(T)}{3}$ is $\mathcal{N}(0,T)$ distributed, hence $S(T)$ is $\mathcal{N}(0, 9T)$ distributed. Therefore,

$$\begin{aligned}
C(0) &= \widetilde{\mathbb{E}}\Big[(S(T) - K)^+\Big] \\
&= \int_{-\infty}^{\infty} (x-K)^+ \frac{1}{\sqrt{18\pi T}} e^{-\frac{x^2}{18T}}\,dx \\
&= \int_{K}^{\infty} (x-K) \frac{1}{\sqrt{18\pi T}} e^{-\frac{x^2}{18T}}\,dx \\
&= \int_{K/3\sqrt{T}}^{\infty} (3\sqrt{T}\,y - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy \\
&= \int_{K/3\sqrt{T}}^{\infty} 3\sqrt{T}\,y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy - \int_{K/3\sqrt{T}}^{\infty} K \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy \\
&= e^{-K^2/18T} 3\sqrt{\frac{T}{2\pi}} - K\Big(1 - N(K/3\sqrt{T})\Big),
\end{aligned}$$

where $N(y)$ is the standard normal distribution function.

(4) Let $\{(W_1(t), W_2(t)) : t \ge 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \ge 0\}$ and $\{S_2(t) : t \ge 0\}$ with corresponding SDE given by

$$\begin{aligned}
dS_1(t) &= 2S_1(t)\,dW_1(t) + 3S_1(t)\,dW_2(t) \\
dS_2(t) &= S_2(t)\,dt + S_2(t)\,dW_1(t),
\end{aligned}$$

(a) Show that $\{S_1(t)S_2(t) : t \ge 0\}$ is a 2-dimensional Itô-process. (1 pt)

(b) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\widetilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \le t \le T\}$. (1 pt)

Proof (a) : We apply Itô product rule, we have

$$d(S_1(t)S_2(t)) = S_1(t)\,dS_2(t) + S_2(t)\,dS_1(t) + dS_1(t)\,dS_2(t).$$

Using $dS_1(t) = 2S_1(t)\,dW_1(t) + 3S_1(t)\,dW_2(t)$, $dS_2(t) = S_2(t)\,dt + S_2(t)\,dW_1(t)$ and $dS_1(t)dS_2(t) = 2S_1(t)S_2(t)\,dt$, we get after simplifying,

$$d(S_1(t)S_2(t) = 3S_1(t)S_2(t)\,dt + 3S_1(t)S_2(t)\,dW_1(t) + 3S_1(t)S_2(t)\,dW_2(t).$$

Equivalently,

$$\begin{aligned}
S_1(t)S_2(t) = S_1(0)S_2(0) &+ \int_0^t 3S_1(u)S_2(u)\,du + \int_0^t 3S_1(u)S_2(u)\,dW_1(u) \\
&+ \int_0^t 3S_1(u)S_2(u)\,dW_2(u).
\end{aligned}$$

Hence, $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô process.

Proof (b) : Using the notation of the book, we have $\alpha_1 = 0$, $\sigma_{11} = 2$, $\sigma_{12} = 3$, $\alpha_2 = 1$, $\sigma_{21} = 1$, $\sigma_{22} = 0$. The market price equations in this case is the system,

$$\begin{aligned} -r &= 2\theta_1(t) + 3\theta_2(t) \\ 1 - r &= \theta_1(t). \end{aligned}$$

Solving for $\theta_1(t), \theta_2(t)$, we get

$$\begin{aligned} \theta_1(t) &= 1 - r \\ \theta_2(t) &= \frac{r-2}{3}. \end{aligned}$$

Setting

$$\begin{aligned} Z &= \exp\Big\{ -\int_0^T \Big(\theta_1(t)\, dW_1(t) + \theta_2(t)\, dW_2(t)\Big) - \frac{1}{2}\int_0^T \Big(\theta_1^2(t) + \theta_2^2(t)\Big)\, dt\Big\} \\ &= \exp\Big\{(r-1)W_1(T) + \frac{2-r}{3}W_2(T) - \frac{1}{2}\Big((1-r)^2 + \frac{(r-2)^2}{9}\Big)T\Big\}, \end{aligned}$$

the risk-neutral measure is given by $\widetilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}$. To check this, we set $\widetilde{W}_1(t) = (1-r)t + W_1(t)$ and $\widetilde{W}_2(t) = \frac{r-2}{3}t + W_2(t)$, By the 2-dimensional Girsanov Theorem the process $\{(\widetilde{W}_1(t), \widetilde{W}_2(t) : 0 \leq t \leq T\}$ is a 2-dimensional Brownian motion under $\widetilde{\mathbb{P}}$. Rewriting $e^{-rt}S_1(t), e^{-rt}S_2(t)$ in terms of $\widetilde{W}_1(t), \widetilde{W}_1(t)$, we get after applying Itô product rule

$$\begin{aligned} d(e^{-rt}S_1(t)) &= e^{-rt}S_1(t)(2d\widetilde{W}_1(t) + 3d\widetilde{W}_2(t)) \\ d(e^{-rt}S_2(t)) &= e^{-rt}S_2(t)d\widetilde{W}_1(t), \end{aligned}$$

which shows that the discounted price processes are Itô integrals and hence martingales under $\widetilde{\mathbb{P}}$.