Solutions Retake Exam: Inleiding Financiële Wiskunde 2018-2019

(1) Let \((W(t) : t \geq 0)\) be a Brownian motion, define a process \((X(t) : t \geq 0)\) by \(X(t) = \frac{1}{\sqrt{2}}W(2t)\).

(a) Prove that \((X(t) : t \geq 0)\) is a Brownian motion. (1 pt)

(b) Let \(Y(t) = X^2(t) - 2\sqrt{t}\) for some non-negative constant \(c\) and for all \(t \geq 0\). For which value of \(c\) is the process \((Y(t) : t \geq 0)\) a martingale with respect to the filtration \((\mathcal{F}(t) : t \geq 0)\) with \(\mathcal{F}(t) = \sigma(X(s) : s \leq t)\). (1 pt)

(c) Consider the process \(\{Z(t) : t \geq 0\}\) defined by \(Z(t) = \int_0^t e^u dX(u)\). Determine the distribution of \(Z(t)\) and calculate \(\mathbb{P}(Z(t) \leq 1)\). (1 pt)

**Proof (a):** We check that the process \((X(t) : t \geq 0)\) satisfies all the properties of a Brownian motion. We have

(i) \(X(0) = \frac{1}{\sqrt{2}}W(0) = 0\).

(ii) Since \((W(t) : t \geq 0)\) has continuous paths and the function \(t \to 2tt\) is continuous and so is multiplication by \(1/\sqrt{2}\), we see that the process \((X(t) : t \geq 0)\) also has continuous paths.

(iii) If \(0 \leq u < v < s < t\), then clearly \(0 \leq 2u < 2v < 2s < 2t\). Hence \(W(2v) - W(2u)\) and \(W(2t) - W(2s)\) are independent implying that \(X(v) - X(u)\) and \(X(t) - X(s)\) are independent. Therefore the process \((X(t) : t \geq 0)\) has independent increment.

(iv) if \(s < t\), then \(W(2t) - W(s)\) is normally distributed with mean zero and variance \(2t - 2s = 2(t - s)\). hence \(X(t) - X(s)\) is also normally distributed with mean

\[
\mathbb{E}(X(t) - X(s)) = \frac{1}{\sqrt{2}}\mathbb{E}(W(2t) - W(2s)) = 0,
\]

and variance

\[
\text{Var}(X(t) - X(s)) = \frac{1}{2}\text{Var}(W(2t) - W(2s)) = (t - s).
\]

Therefore, \((X(t) : t \geq 0)\) is a Brownian motion.

**Proof (b):** The underlying filtration is given by \(\mathcal{F}(t) = \sigma(X(s) : s \leq t)\). Now let \(s < t\), and note that \(Y(t) - Y(s)\) is independent of \(\mathcal{F}(s)\) while \(Y(s)\) is \(\mathcal{F}(s)\)-measurable. Hence,

\[
\mathbb{E}[Y(t)|\mathcal{F}(s)] = \mathbb{E}[Y(t) - Y(s) + Y(s)|\mathcal{F}(s)]
= \mathbb{E}[Y(t) - Y(s)] + Y(s)
= \mathbb{E}[X^2(t) - X^2(s)] - 2\sqrt{c}(t - s) + Y(s)
= \mathbb{E}[\frac{1}{2}W^2(2t) - \frac{1}{2}W^2(2s)] - 2\sqrt{c}(t - s) + Y(s)
= \frac{1}{2}(2t - 2s) - 2\sqrt{c}(t - s) + Y(s)
= (1 - 2\sqrt{c})(t - s) + Y(s).
\]

Hence, for the process \((Y(t) : t \geq 0)\) to be a martingale, we must have \(c = 1/4\).
Proof (c): Note that $Z(t)$ is an Itô-integral of a deterministic process, hence it is normally distributed with mean 0 and variance
\[
\Var \left( \int_0^t e^u \, dW(u) \right) = \int_0^t e^{2u} \, du = \frac{1}{2}(e^{2t} - 1).
\]
Thus,
\[
P(Z(t) \leq 1) = \mathbb{P} \left( \frac{Z(t)}{\sqrt{\frac{1}{2}(e^{2t} - 1)}} \leq \frac{1}{\sqrt{\frac{1}{2}(e^{2t} - 1)}} \right) = N \left( \frac{1}{\sqrt{\frac{1}{2}(e^{2t} - 1)}} \right),
\]
where $N(x)$ is the standard normal distribution function.

(2) Let $(W(t) : t \geq 0)$ be a Brownian motion and let $\{\mathcal{F}(t) : t \geq 0\}$ be its natural filtration. Consider the Itô process $\{X(t) : t \geq 0\}$ with
\[
X(t) = X(0) + \int_0^t \alpha X(u) \, du + \int_0^t 2X(u) \, dW(u)
\]
with $\alpha$ and $X(0)$ some constants.

(a) Show that $\mathbb{E}[X(t)] = X(0)e^{\alpha t}$. (Hint: you are allowed to interchange the integral and the expectation) (1 pt)

(b) Show that the process $\{X^2(t) : t \geq 0\}$ is an Itô process. (1 pt)

(c) For which values of $\alpha$ is the process $\{X^2(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$? (1 pt)

Proof (a): Since an Itô integral has zero expectation, we see that
\[
\mathbb{E}[X(t)] = X(0) + \mathbb{E} \left[ \int_0^t \alpha X(u) \, du \right] = X(0) + \int_0^t \alpha \mathbb{E}[X(u)] \, du.
\]
Let $m(t) = \mathbb{E}[X(t)]$, then the above equation reads $m(t) = X(0) + \int_0^t \alpha m(u) \, du$. Equivalently,
\[
\frac{dm(t)}{dt} = \alpha m(t)
\]
which has solution $\mathbb{E}[X(t)] = m(t) = X(0)e^{\alpha t}$.

Proof (b): We apply Itô Doeblin formula to the function $f(x) = x^2$. We have $f_x(x) = 2x$, $f_{xx}(x) = 2$ and $dX(t)dX(t) = 4X^2(t) \, dt$. Hence,
\[
\begin{align*}
dX^2(t) &= df(X(t)) = 2X(t) \, dX(t) + 4X^2(t) \, dt \\
&= 2X(t)[\alpha X(t) \, dt + 2X(t) \, dW(t)] + 4X^2(t) \, dt \\
&= (2\alpha + 4)X^2(t) \, dt + 4X^2(t) \, dW(t).
\end{align*}
\]
Equivalently,
\[
X^2(t) = X^2(0) + \int_0^t (2\alpha + 4)X^2(u) \, du + \int_0^t 4X^2(u) \, dW(u).
\]
Therefore, $\{X^2(t) : t \geq 0\}$ is an Itô process.

Proof (c): Since Itô integrals are martingales, from part (b) we see that $\{X^2(t) : t \geq 0\}$ is a martingale if $2\alpha + 4 = 0$, which implies that $\alpha = -2$.

(3) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with $S(t) = t^3 + 3W(t)$.

(a) Construct a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0$, $A \in \mathcal{F}$) such that the price process $\{S(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. (1 pt)

(b) Consider a European call option on this stock with expiration date $T$ and strike price $K$. Find an expression for $C(0) = \mathbb{E} \left[ (S(T) - K)^+ \right]$, the price of this option at time 0. (1 pt)
Proof (a): Define \( \theta(t) = t^2 \), then \( S(t)/3 = \int_0^t \theta(u) \, du + W(t) \). Consider the random variable \( Z \) defined by

\[
Z = \exp \left( -\int_0^T \theta(u) \, dW(u) - \frac{1}{2} \int_0^T \theta^2(u) \, du \right) = \exp \left( -\int_0^T t^2 \, dW(u) - \frac{1}{10} t^3 \right).
\]

Note that \( \int_0^t \theta(u) \, dW(u) \), \( \int_0^t \theta^2(u) \, du \) and \( \theta \) are continuous functions on the compact interval \([0, T]\), hence they are all bounded. This implies that \( \mathbb{E} \left[ \int_0^T \theta^2(u)Z^2(u) \, du \right] < \infty \). Define the measure \( \tilde{\mathbb{P}} \) on \( \mathcal{F} \) by \( \tilde{\mathbb{P}}(A) = \int_A Z \, d\mathbb{P} \). By Girsanov’s Theorem, the process \( \{ \frac{S(t)}{3} \} \) is a Brownian motion under \( \tilde{\mathbb{P}} \) and hence is a martingale under \( \tilde{\mathbb{P}} \). Therefore, \( \{ S(t) : 0 \leq t \leq T \} \) is a martingale under \( \tilde{\mathbb{P}} \).

Proof (b): By part (a), we see that under the measure \( \tilde{\mathbb{P}} \), the random variable \( \frac{S(T)}{3} \) is \( \mathcal{N}(0, T) \) distributed, hence \( S(T) \) is \( \mathcal{N}(0, 9T) \) distributed. Therefore,

\[
C(0) = \mathbb{E} \left[ (S(T) - K)^+ \right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{18\pi T}} e^{-\frac{x^2}{18T}} \, dx = \int_K^{\infty} \frac{1}{\sqrt{18\pi T}} e^{-\frac{x^2}{18T}} \, dx
= \int_{3\sqrt{T}/2}^{\infty} \left( 3\sqrt{T}y - K \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy
= \int_{3\sqrt{T}/2}^{\infty} 3\sqrt{T}y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy - \int_{3\sqrt{T}/2}^{\infty} K \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy
= e^{-K^2/18T} \left[ \frac{T}{2\pi} - K \left( 1 - N(3\sqrt{T}/2) \right) \right],
\]

where \( N(y) \) is the standard normal distribution function.

(4) Let \( \{(W_1(t), W_2(t)) : t \geq 0\} \) be a 2-dimensional Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Consider two price processes \( \{ S_1(t) : t \geq 0 \} \) and \( \{ S_2(t) : t \geq 0 \} \) with corresponding SDE given by

\[
dS_1(t) = 2S_1(t) \, dW_1(t) + 3S_1(t) \, dW_2(t)
dS_2(t) = S_2(t) \, dt + S_2(t) \, dW_1(t),
\]

(a) Show that \( \{ S_1(t)S_2(t) : t \geq 0 \} \) is a 2-dimensional Itô-process. (1 pt)

(b) Consider a finite time \( T \) (expiration date), and suppose the interest rate is a constant, i.e. \( R(t) = r \) for all \( t > 0 \). Show that the market price equations have a unique solution, and determine the risk-neutral probability measure \( \tilde{\mathbb{P}} \) for the process \( \{ (S_1(t), S_2(t) : 0 \leq t \leq T \} \). (1 pt)

Proof (a): We apply Itô product rule, we have

\[
d(S_1(t)S_2(t)) = S_1(t) \, dS_2(t) + S_2(t) \, dS_1(t) + dS_1(t) \, dS_2(t).
\]

Using \( dS_1(t) = 2S_1(t) \, dW_1(t) + 3S_1(t) \, dW_2(t) \), \( dS_2(t) = S_2(t) \, dt + S_2(t) \, dW_1(t) \) and \( dS_1(t) \, dS_2(t) = 2S_1(t)S_2(t) \, dt \), we get after simplifying,

\[
d(S_1(t)S_2(t)) = 3S_1(t)S_2(t) \, dt + 3S_1(t)S_2(t) \, dW_1(t) + 3S_1(t)S_2(t) \, dW_2(t).
\]

Equivalently,

\[
S_1(t)S_2(t) = S_1(0)S_2(0) + \int_0^t 3S_1(u)S_2(u) \, du + \int_0^t 3S_1(u)S_2(u) \, dW_1(u)
+ \int_0^t 3S_1(u)S_2(u) \, dW_2(u).
\]
Hence, \( \{S_1(t)S_2(t) : t \geq 0 \} \) is a 2-dimensional Itô process.

**Proof (b):** Using the notation of the book, we have \( \alpha_1 = 0, \sigma_{11} = 2, \sigma_{12} = 3, \alpha_2 = 1, \sigma_{21} = 1, \sigma_{22} = 0 \). The market price equations in this case is the system,

\[
-r = 2\theta_1(t) + 3\theta_2(t) \\
1 - r = \theta_1(t).
\]

Solving for \( \theta_1(t), \theta_2(t) \), we get

\[
\theta_1(t) = 1 - r \\
\theta_2(t) = \frac{r - 2}{3}.
\]

Setting

\[
Z = \exp \left\{ - \int_0^T \left( \theta_1(t) \, dW_1(t) + \theta_2(t) \, dW_2(t) \right) - \frac{1}{2} \int_0^T \left( \theta_1^2(t) + \theta_2^2(t) \right) \, dt \right\}
\]

\[
= \exp \left\{ (r - 1)W_1(T) + \frac{2 - r}{3} W_2(T) - \frac{1}{2} \left( (1 - r)^2 + \frac{(r - 2)^2}{9} \right) T \right\},
\]

the risk-neutral measure is given by \( \widetilde{P}(A) = \int_A Z \, d\overline{P} \). To check this, we set \( \widetilde{W}_1(t) = (1 - r)t + W_1(t) \) and \( \widetilde{W}_2(t) = \frac{r - 2}{3} t + W_2(t) \), By the 2-dimensional Girsanov Theorem the process \( \{\widetilde{W}_1(t), \widetilde{W}_2(t) : 0 \leq t \leq T\} \) is a 2-dimensional Brownian motion under \( \overline{P} \). Rewriting \( e^{-rt}S_1(t), e^{-rt}S_2(t) \) in terms of \( \widetilde{W}_1(t), \widetilde{W}_1(t) \), we get after applying Itô product rule

\[
d(e^{-rt}S_1(t)) = e^{-rt}S_1(t)(2d\widetilde{W}_1(t) + 3d\widetilde{W}_2(t)) \\
d(e^{-rt}S_2(t)) = e^{-rt}S_2(t)d\widetilde{W}_1(t),
\]

which shows that the discounted price processes are Itô integrals and hence martingales under \( \overline{P} \).