Hertentamen: Inleiding Financiele Wiskunde 2017-2018

(1) Consider a 2-period binomial model with \( S_0 = 10 \), \( u = 1.2 \), \( d = 0.8 \), and \( r = 0.1 \). Suppose the real probability measure \( \mathbb{P} \) satisfies \( \mathbb{P}(H) = p = \frac{1}{2} = \mathbb{P}(T) \).

(a) Consider a European option with payoff \( V_2 = \max((S_0, S_1, S_2) - 10)^+ \). Determine the price \( V_n \) at time \( n = 0, 1, 2 \). (0.75 pt)

(b) Consider the utility function \( U(x) = \ln(2x + 1) \) \( (x > 0) \). Show that the random variable \( X = X_2 \) (which is a function of the two coin tosses) that maximizes \( \mathbb{E}(U(X)) \) subject to the condition that \( \tilde{\mathbb{E}}(1 + r)^2 X_0 = X_0 \) is given by

\[
X = X_2 = \frac{1}{Z} \left( (1.1)^2 X_0 + \frac{1}{2} \right) - \frac{1}{2}.
\]

(1 pt)

(c) Consider part (b) and assume \( X_0 = 100 \). Determine the value of the optimal portfolio process \( \{\Delta_0, \Delta_1\} \) and the value of the corresponding wealth process \( \{X_0, X_1, X_2\} \). (1.25 pt)

(d) Consider now an Asian American put option with expiration \( N = 2 \), and intrinsic value \( G_n = 12 - \max(S_0, \ldots, S_n), n = 0, 1, 2 \). Determine the price \( V_n \) at time \( n = 0, 1, 2 \) of the American option. Find the optimal exercise time \( \tau^*(\omega_1, \omega_2) \) for all \( \omega_1, \omega_2 \). (1 pt)

(2) Consider an \( N \)-period binomial model with real probability measure \( \mathbb{P} \) satisfying \( \mathbb{P}(H) = p \) and \( q = 1 - p = \mathbb{P}(T) \). For \( n = 1, \ldots, N \) define

\[
Y_n = \begin{cases} 
2, & \text{if } \omega_n = H, \\
-3, & \text{if } \omega_n = T.
\end{cases}
\]

Set \( M_0 = 0 \) and let \( M_n = \sum_{i=1}^{n} Y_i, n = 1, \ldots, N \).

(a) Prove that

\[
\mathbb{P}(M_n = k) = \begin{cases} 
\left( \frac{n}{3n+k} \right)^{p(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n + k \equiv 0 \mod 5, \\
0, & \text{otherwise}.
\end{cases}
\]

(1 pt)

(b) Define \( U_n = M_n Y_n \) for \( n = 1, \ldots, N \). Show that for any function \( f : \mathbb{R}^2 \to \mathbb{R} \), and any \( n = 0, 1, \ldots, N \),

\[
\mathbb{E}_n \left( f(U_{n+1}, Y_{n+1}) \right) = pf \left( \frac{2U_n}{Y_n} + 4, 2 \right) + qf \left( \frac{-3U_n}{Y_n} + 9, -3 \right).
\]

Conclude that \( (U_1, Y_1, \ldots, U_N, Y_N) \) is a Markov process under \( \mathbb{P} \). (1.25 pt)

(c) For which values of \( p \) is the process \( \{M_n : n = 0, 1, \ldots, N\} \) a (i) martingale, (ii) submartingale, (iii) supermartingale? (1 pt)
(d) Suppose \( p = \mathbb{P}(H) = 3/5 \) and \( q = 1 - p = \mathbb{P}(T) = 2/5 \). Define the stopping time \( \tau \) by
\[
\tau = \inf\{n \geq 0 : M_n = 3\}.
\]
Determine the value of \( \mathbb{E}(M_{n \wedge \tau}) \) for \( n = 0, 1, \cdots, N \). (0.5 pt)

(3) Consider the (infinite) binomial model with up factor \( u = \sqrt{2} \), down factor \( d = \frac{1}{\sqrt{2}} \) and interest rate \( r = \frac{3\sqrt{2}}{4} - 1 \). Suppose the real probability \( \mathbb{P} \) is given by \( \mathbb{P}(H) = p = 2/3 \) and \( \mathbb{P}(T) = q = 1/3 \).

Define the process \( (M_n : n = 0, 1, \cdots) \) by \( M_0 = 0 \) and \( M_n = \sum_{i=1}^{n} X_i \) where
\[
X_i = \begin{cases} 
1, & \text{if } \omega_i = H, \\
-1, & \text{if } \omega_i = T,
\end{cases}
\]
Consider a perpetual American put option with \( S_0 = 4 \) and strike price \( K = 8 \)

(a) Show that the price process \( (S_n : n = 0, 1, \cdots) \) is given by \( S_n = 2^{2 + \frac{1}{2}M_n} \), and the risk-neutral probability \( \mathbb{P} \) is given by \( \mathbb{P}(H) = \bar{p} = 1/2 = \bar{q} = \mathbb{P}(T) \). (1 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 2 euros. What is then the price at time 0 of such an option? (0.75 pt)

(c) Determine under \( \mathbb{P} \), the probability that the price reaches \( 4\sqrt{2} \) euros for the first time at time \( n = 5 \)? (0.5 pt)