Hertentamen: Inleiding Finaniele Wiskunde 2017-2018

(1) Consider a 2-period binomial model with $S_0 = 10$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = \mathbb{P}(T)$.

(a) Consider a European option with payoff $V_2 = \max\left((S_0, S_1, S_2) - 10\right)^+$. Determine the price V_n at time $n = 0, 1, 2$. (0.75 pt)

(b) Consider the utility function $U(x) = \ln(2x+1)$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $\mathbb{E}(U(X))$ subject to the condition that $\widetilde{\mathbb{E}}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

$$X = X_2 = \frac{1}{Z}\left[(1.1)^2 X_0 + \frac{1}{2}\right] - \frac{1}{2}.$$

(1 pt)

(c) Consider part (b) and assume $X_0 = 100$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$. (1.25 pt)

(d) Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 12 - \max\Big(S_0, \cdots, S_n\Big)$, $n = 0, 1, 2$. Determine the price V_n at time $n = 0, 1, 2$ of the American option. Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$. (1 pt)

(2) Consider an N-period binomial model with real probability measure $\mathbb{P}$ satisfisfying $\mathbb{P}(H) = p$ and $q = 1 - p = \mathbb{P}(T)$. For $n = 1, \cdots, N$ define

$$Y_n = \begin{cases} 2, & \text{if } \omega_n = H, \\ -3, & \text{if } \omega_n = T. \end{cases}$$

Set $M_0 = 0$ and let $M_n = \sum_{i=1}^{n} Y_i$, $n = 1, \cdots, N$.

(a) Prove that

$$\mathbb{P}(M_n = k) = \begin{cases} \binom{n}{\frac{3n+k}{5}} p^{(3n+k)/5} q^{(2n-k)/5}, & \text{if } 3n + k \equiv 0 \bmod 5, \\ 0, & \text{otherwise }. \end{cases}$$

(1 pt)

(b) Define $U_n = M_n Y_n$ for $n = 1, \cdots, N$. Show that for any function $f : \mathbb{R}^2 \to \mathbb{R}$, and any $n = 0, 1, \cdots, N$,

$$\mathbb{E}_n\Big(f(U_{n+1}, Y_{n+1})\Big) = pf\Big(\frac{2U_n}{Y_n} + 4, 2\Big) + qf\Big(\frac{-3U_n}{Y_n} + 9, -3\Big).$$

Conclude that $(U_1, Y_1), \cdots, (U_N, Y_N)$ is a Markov process under $\mathbb{P}$. (1.25 pt)

(c) For which values of p is the process $\{M_n : n = 0, 1, \cdots, N\}$ a (i) martingale, (ii) submartingale, (iii) supermartingale? (1 pt)

(d) Suppose $p = \mathbb{P}(H) = 3/5$ and $q = 1 - p = \mathbb{P}(T) = 2/5$. Define the stopping time τ by

$$\tau = \inf\{n \geq 0 : M_n = 3\}.$$

Determine the value of $\mathbb{E}(M_{n\wedge\tau})$ for $n = 0, 1, \cdots, N$. (0.5 pt)

(3) Consider the (infinite) binomial model with up factor $u = \sqrt{2}$, down factor $d = \dfrac{1}{\sqrt{2}}$ and interest rate $r = \dfrac{3\sqrt{2}}{4} - 1$. Suppose the real probability $\mathbb{P}$ is given by $\mathbb{P}(H) = p = 2/3$ and $\mathbb{P}(T) = q = 1/3$. Define the process $(M_n : n = 0, 1, \cdots)$ by $M_0 = 0$ and $M_n = \sum_{i=1}^{n} X_i$ where

$$X_i = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

Consider a perpetual American put option with $S_0 = 4$ and strike price $K = 8$

(a) Show that the price process $(S_n : n = 0, 1, \cdots)$ is given by $S_n = 2^{2+\frac{1}{2}M_n}$, and the risk-neutral probability $\widetilde{\mathbb{P}}$ is given by $\widetilde{\mathbb{P}}(H) = \widetilde{p} = 1/2 = \widetilde{q} = \widetilde{\mathbb{P}}(T)$. (1 pt)

(b) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 2 euros. What is then the price at time 0 of such an option? (0.75 pt)

(c) Determine under $\mathbb{P}$, the probability that the price reaches $4\sqrt{2}$ euros for the first time at time $n = 5$? (0.5 pt)