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Retake Exam: Inleiding Financiele Wiskunde 2018-2019

(1) Let $(W(t) : t \geq 0)$ be a Brownian motion, define a process $(X(t) : t \geq 0)$ by $X(t) = \frac{1}{\sqrt{2}} W(2t)$.

(a) Prove that $(X(t) : t \geq 0)$ is a Brownian motion. (1 pt)

(b) Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $(Y(t) : t \geq 0)$ a martingale with respect to the filtration $(\mathcal{F}(t) : t \geq 0)$ with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$. (1 pt)

(c) Consider the process $\{Z(t) : t \geq 0\}$ defined by $Z(t) = \int_0^t e^u \, dX(u)$. Determine the distribution of $Z(t)$ and calculate $\mathbb{P}(Z(t) \leq 1)$. (1 pt)

(2) Let $(W(t) : t \geq 0)$ be a Brownian motion and let $\{\mathcal{F}(t) : t \geq 0\}$ be its natural filtration. Consider the Itô process $\{X(t) : t \geq 0\}$ with

$$X(t) = X(0) + \int_0^t \alpha X(u) \, du + \int_0^t 2X(u) \, dW(u)$$

with α and $X(0)$ some constants.

(a) Show that $\mathbb{E}[X(t)] = X(0)e^{\alpha t}$. (Hint: you are allowed to interchange the integral and the expectation) (1 pt)

(b) Show that the process $\{X^2(t) : t \geq 0\}$ is an Itô process. (1 pt)

(c) For which values of α is the process $\{X^2(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$? (1 pt)

(3) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with $S(t) = t^3 + 3W(t)$.

(a) Construct a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\widetilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0$, $A \in \mathcal{F}$) such that the price process $\{S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. (1 pt)

(b) Consider a European call option on this stock with expiration date T and strike price K. Find an expression for $C(0) = \widetilde{\mathbb{E}}\Big[(S(T) - K)^+\Big]$, the price of this option at time 0. (1 pt)

(4) Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by

$$dS_1(t) = 2S_1(t) \, dW_1(t) + 3S_1(t) \, dW_2(t)$$
$$dS_2(t) = S_2(t) \, dt + S_2(t) \, dW_1(t),$$

(a) Show that $\{S_1(t)S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (1 pt)

(b) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\widetilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (1 pt)