Uitwerkingen Deeltentamen 1: Inleiding Finaniele Wiskunde 2017-2018

(1) Consider an 2-period binomial model with $S_0 = 200$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = \mathbb{P}(T)$. Denote the risk-neutral probability (as usual) by $\widetilde{\mathbb{P}}$.

(a) Consider an option with payoff $V_2 = \left(\frac{1}{4}S_1 + \frac{3}{4}S_2 - 180\right)^+$. Determine the price V_n at time $n = 0, 1$. (1 pt)

(b) Suppose an investor owns the option and intends to hold it until the expiration date ($N = 2$) and receives the payoff V_2. So, at time 0 the investor has a capital of V_0 (your answer obtained in part (a)) which is tied up to the option and wants to earn the interest rate of 10% on this capital until time 2 without investing anymore money, and regardless of how the coin tossing turn out, the investor wants to have $(1.1)^2 V_0$. Determine a strategy (i.e. a wealth process) made of stocks and money market that accomplishes this goal. (2 pts)

(c) Consider the Radon-Nikodym derivative process Z_0, Z_1, Z_2, where $Z_n = \mathbb{E}_n(Z)$ and $Z(\omega) = Z(\omega_1\omega_2\omega_3) = \frac{\widetilde{\mathbb{P}}(\omega_1\omega_2\omega_3)}{\mathbb{P}(\omega_1\omega_2\omega_3)}$. Determine explicitly the values of $Z_n(\omega)$ for $n = 0, 1, 2$ and all $\omega = (\omega_1, \omega_2, \omega_3)$. (1 pt)

Solution (a): The risk neutral measure $\widetilde{\mathbb{P}}$ is given by $\widetilde{\mathbb{P}}(H) = \widetilde{p} = \frac{1 + r - d}{u - d} = \frac{3}{4} = 0.75$, and $\widetilde{\mathbb{P}}(T) = \widetilde{q} = \frac{1}{4} = 0.25$. From the hypothesis, we have

$$V_2(HH) = 96,\ V_2(HT) = 24,\ V_2(TH) = 4,\ V_2(TT) = 0.$$

Using Theorem 1.2.2 we have,

$$\begin{aligned} V_1(H) &= \frac{1}{1.1}\Big[(0.75)(96) + (0.25)(24)\Big] = 70.91, \\ V_1(T) &= \frac{1}{1.1}\Big[(0.75)(4) + (0.25)(0)\Big] = 2.73, \\ V_0 &= \frac{1}{1.1}\Big[(0.75)(70.91) + (0.25)(2.73)\Big] = 48.97. \end{aligned}$$

Solution (b): Let us denote the corresponding wealth process, as given in Theorem 1.2.2, associated to the option price process V_0, V_1, V_2 by X_0, X_1, X_2. Note that $V_0 = X_0$ and

$$V_i = X_i = \Delta_{i-1}S_i + (1.1)(X_{i-1} - \Delta_{i-1}S_{i-1}) \text{ for } i = 1, 2.$$

The goal is to construct a strategy (wealth process) X_0', X_1', X_2' with

$$\begin{aligned} X_0' &= 0, \\ X_1' &= \Delta_0' S_1 + (1.1)(X_0' - \Delta_0' S_0) = \Delta_0' S_0 + (1.1)(-\Delta_0' S_0), \\ X_2' &= \Delta_1' S_2 + (1.1)(X_1' - \Delta_1' S_1) \end{aligned}$$

and satisfying

$$X_1' + X_1 = (1.1)V_0, \text{ and } X_2' + X_2 = (1.1)^2 V_0,$$

where $V_0 = 48.97$ as was found in part (a). Replacing X_1', X_1 by their expression in terms of Δ_0' and Δ_0 respectively in the equation $X_1' + X_1 = (1.1)V_0$, after simplifying one ends up with

$$\Delta_0' = -\Delta_0 = -\frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{70.91 - 2.73}{240 - 160} = -0.85.$$

Similarly replacing X_2', X_2 by their expression in terms of Δ_1' and Δ_1 respectively in the equation $X_2' + X_2 = (1.1)^2 V_0$, and using the fact that $X_1' = (1.1)V_0 - X_1$, one gets after simplification that $\Delta_1' = -\Delta_1$. In particular,

$$\Delta_1'(H) = -\Delta_1(H) = -\frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{96 - 24}{288 - 192} = -0.75,$$

and

$$\Delta_1'(T) = -\Delta_1(T) = -\frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_1(TT)} = \frac{4 - 0}{192 - 128} = -0.0625.$$

Regardless of what the values of $\Delta_1'(H) = -\Delta_1(H)$ and $\Delta_1'(T) = -\Delta_1(T)$ are, at time 2 we have (after replacing $X_1' + X_1 = (1.1)V_0$),

$$X_2' + X_2 = (1.1)^2 V_0 = (1.1)^2(48.97) = 59.25.$$

Solution (c): We have seen that

$$Z_n(\omega) = Z_n(\omega_1, \cdots, \omega_n) = \left(\frac{\widetilde{p}}{p}\right)^{\#H(\omega_1,\cdots,\omega_n)} \left(\frac{\widetilde{q}}{q}\right)^{\#T(\omega_1,\cdots,\omega_n)},$$

where $\#H(\omega_1, \cdots, \omega_n)$ is the number of H in $\omega_1, \cdots, \omega_n$, and similarly for $\#T(\omega_1, \cdots, \omega_n)$. Since $\frac{\widetilde{p}}{p} = \frac{3}{2}$ and $\frac{\widetilde{q}}{q} = \frac{1}{2}$, we have

$$\begin{aligned}
Z_0(\omega) &= Z_0 = 1, \text{ for all } \omega, \\
Z_1(H\omega_2\omega_3) &= Z_1(H) = \frac{3}{2}, \text{ for all } \omega_2, \omega_3 \\
Z_1(T\omega_2\omega_3) &= Z_1(T) = \frac{1}{2}, \text{ for all } \omega_2, \omega_3 \\
Z_2(HH\omega_3) &= Z_2(HH) = \frac{9}{4}, \text{ for all } \omega_3 \\
Z_2(HT\omega_3) &= Z_2(HT) = \frac{3}{4}, \text{ for all } \omega_3 \\
Z_2(TH\omega_3) &= Z_2(TH) = \frac{3}{4}, \text{ for all } \omega_3 \\
Z_2(TT\omega_3) &= Z_2(TT) = \frac{1}{4}, \text{ for all } \omega_3.
\end{aligned}$$

(2) Consider the N-period Binomial model with risk neutral probability measure $\widetilde{\mathbb{P}}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

$$Y_0 = 1, \text{ and } Y_n = \frac{1}{(1 + X_0) \cdots (1 + X_{n-1})}, \; n = 1, \cdots, N.$$

(a) Let $U_n = \widetilde{\mathbb{E}}_n \left[\frac{Y_N}{Y_n}\right]$, $n = 0, 1, \cdots, N$. Show that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$. (1.5 pts)

(b) (Corrected verion) Let U_n be as given in part (a). Set $I_0 = 0$ and define $I_n = \sum_{j=0}^{n-1} Y_{j+1}(Y_{j+1}U_{j+1} - Y_jU_j)$, $n = 1, \cdots, N$. Show that $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$. (1.5 pts)

Solution (a): First note that the process $\{X_n : n = 0, \cdots, N\}$ is adapted, hence the random variable Y_n is known at time $n - 1$, i.e. depends on the first $n - 1$ tosses, $n = 1, \cdots, N$. Hence,

$$U_n = \widetilde{\mathbb{E}}_n \left[\frac{Y_N}{Y_n}\right] = \frac{1}{Y_n} \widetilde{\mathbb{E}}_n[Y_N],$$

which implies $Y_nU_n = \widetilde{\mathbb{E}}_n[Y_N]$. Using the iteration property of conditional expectations, or directly Theorem 3.2.1, one has that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$.

Solution (b): First note that Y_{n+1} is known at time n, and

$$I_{n+1} = I_n + Y_{n+1}(Y_{n+1}U_{n+1} - Y_nU_n).$$

From part (a), we have that $Y_0U_0, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$, and hence $\widetilde{\mathbb{E}}_n(Y_{n+1}U_{n+1} - Y_nU_n) = 0$. Thus,

$$\widetilde{\mathbb{E}}_n(I_{n+1}) = I_n + Y_{n+1}\,\widetilde{\mathbb{E}}_n(Y_{n+1}U_{n+1} - Y_nU_n) = I_n.$$

Therefore, $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$.

(3) Consider the N-period binomial model, with expiration process N, up factor u, down factor d and interst rate r. Let $\widetilde{\mathbb{P}}$ be the risk neutral probability and $\mathbb{P}$ the real probability. We denote by $p = \mathbb{P}(H)$ and $\widetilde{p} = \widetilde{\mathbb{P}}(H)$. Let $S_0, S_1, \cdots, S_N$ be the corresponding price process.

(a) Define $Y_n = \sum_{k=0}^{n} S_k$. Show that the process

$$(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$$

is Markov with respect to $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. (Hint: use the random variables $Z_{n+1} = \dfrac{S_{n+1}}{S_n}$ and the Independence Lemma). (2 pts)

(b) Let $V_N = \left(S_N - \dfrac{Y_N}{N+1}\right)^+$. Show that for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$E_n(ZV_N) = Z_n(1+r)^{N-n} f_n(Y_n, S_n),$$

where Z is the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and $Z_n = \mathbb{E}_n(Z)$, $n = 0, 1, \cdots, N$. (1 pt)

Solution (a): Define $Z_{n+1} = \dfrac{S_{n+1}}{S_n}$ for $n = 0, 1, \cdots, N-1$. Then,

$$Y_{n+1} = Y_n + Z_{n+1}S_n, \text{ and } S_{n+1} = Z_{n+1}S_n.$$

Note that Z_{n+1} is independent of the first n tosses, while Y_n and S_n depend only on the first n tosses. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function, by the Independence Lemma, we have

$$\mathbb{E}_n\left(f(Y_{n+1}, S_{n+1})\right) = \mathbb{E}_n\left(f(Y_n + Z_{n+1}S_n, Z_{n+1}S_n)\right) = g(Y_n, S_n),$$

where

$$g(y, s) = \mathbb{E}(f(y + Z_{n+1}s, Z_{n+1}s)) = pf(y + us, us) + qf(y + ds, ds).$$

A similar calculation shows that

$$\widetilde{\mathbb{E}}_n\left(f(Y_{n+1}, S_{n+1})\right) = \widetilde{\mathbb{E}}_n\left(f(Y_n + Z_{n+1}S_n, Z_{n+1}S_n)\right) = h(Y_n, S_n),$$

where

$$h(y, s) = \widetilde{\mathbb{E}}(f(y + Z_{n+1}s, Z_{n+1}s)) = \widetilde{p}f(y + us, us) + \widetilde{q}f(y + ds, ds).$$

Hence, the process

$$(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$$

is Markov with respect to $\mathbb{P}$ and $\widetilde{\mathbb{P}}$.

Solution (b): Let $f(y, s) = \left(s - y(n+1)^{-1}\right)^+$, then $V_N = f(Y_N, S_N)$. Since $(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$ is Markov with respect to $\widetilde{\mathbb{P}}$, by Theorem 2.5.8, for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$V_n = \widetilde{\mathbb{E}}_n(V_N(1+r)^{-(N-n)}) = f_n(Y_n, S_n),$$

(note that $f = f_N$). Thus, by Lemma 3.2.6

$$\mathbb{E}_n(ZV_N) = Z_n\widetilde{\mathbb{E}}_n(V_N) = Z_n\,(1+r)^{N-n} f_n(Y_n, S_n).$$