(1) Consider an 2-period binomial model with $S_0 = 200$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \frac{1}{2} = \mathbb{P}(T)$. Denote the risk-neutral probability (as usual) by $\tilde{\mathbb{P}}$.

(a) Consider an option with payoff $V_2 = \left(\frac{1}{4}S_1 + \frac{3}{4}S_2 - 180\right)^+$. Determine the price $V_n$ at time $n = 0, 1$. (1 pt)

(b) Suppose an investor owns the option and intends to hold it until the expiration date ($N = 2$) and receives the payoff $V_2$. So, at time 0 the investor has a capital of $V_0$ (your answer obtained in part (a)) which is tied up to the option and wants to earn the interest rate of 10% on this capital until time 2 without investing anymore money, and regardless of how the coin tossing turn out, the investor wants to have $(1.1)^2V_0$. Determine a strategy (i.e. a wealth process) made of stocks and money market that accomplishes this goal. (2 pts)

(c) Consider the Radon-Nikodym derivative process $Z_0, Z_1, Z_2$, where $Z_n = \mathbb{E}_n(Z)$ and $Z(\omega) = Z(\omega_1\omega_2\omega_3) = \frac{\tilde{\mathbb{P}}(\omega_1\omega_2\omega_3)}{\tilde{\mathbb{P}}(\omega_1\omega_2\omega_3)}$. Determine explicitly the values of $Z_n(\omega)$ for $n = 0, 1, 2$ and all $\omega = (\omega_1, \omega_2, \omega_3)$. (1 pt)

**Solution (a):** The risk neutral measure $\tilde{\mathbb{P}}$ is given by $\tilde{\mathbb{P}}(H) = \tilde{p} = \frac{1 + r - d}{u - d} = \frac{3}{4} = 0.75$, and $\tilde{\mathbb{P}}(T) = \tilde{q} = \frac{1}{4} = 0.25$. From the hypothesis, we have

\[
V_2(HH) = 96, \quad V_2(HT) = 24, \quad V_2(TH) = 4, \quad V_2(TT) = 0.
\]

Using Theorem 1.2.2 we have,

\[
V_1(H) = \frac{1}{1.1} \left[ (0.75)(96) + (0.25)(24) \right] = 70.91,
\]

\[
V_1(T) = \frac{1}{1.1} \left[ (0.75)(4) + (0.25)(0) \right] = 2.73,
\]

\[
V_0 = \frac{1}{1.1} \left[ (0.75)(70.91) + (0.25)(2.73) \right] = 48.97.
\]

**Solution (b):** Let us denote the corresponding wealth process, as given in Theorem 1.2.2, associated to the option price process $V_0, V_1, V_2$ by $X_0, X_1, X_2$. Note that $V_0 = X_0$ and $V_i = X_i = \Delta_{i-1}S_i + (1.1)(X_{i-1} - \Delta_{i-1}S_{i-1})$ for $i = 1, 2$.

The goal is to construct a strategy (wealth process) $X'_0, X'_1, X'_2$ with

\[
X'_0 = 0,
\]

\[
X'_1 = \Delta'_0S_1 + (1.1)(X'_0 - \Delta'_0S_0) = \Delta'_0S_0 + (1.1)(-\Delta'_0S_0),
\]

\[
X'_2 = \Delta'_1S_2 + (1.1)(X'_1 - \Delta'_1S_1)
\]

and satisfying

\[
X'_1 + X_1 = (1.1)V_0, \quad X'_2 + X_2 = (1.1)^2V_0,
\]

where $V_0 = 48.97$ as was found in part (a). Replacing $X'_1, X_1$ by their expression in terms of $\Delta'_0$ and $\Delta_0$ respectively in the equation $X'_1 + X_1 = (1.1)V_0$, after simplifying one ends up with

\[
\Delta'_0 = -\Delta_0 = \frac{-V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{70.91 - 2.73}{240 - 160} = -0.85.
\]
Similarly replacing $X_2', X_2$ by their expression in terms of $\Delta_1'$ and $\Delta_1$ respectively in the equation $X_2' + X_2 = (1.1)^2 V_0$, and using the fact that $X_1' = (1.1)V_0 - X_1$, one gets after simplification that $\Delta_1' = -\Delta_1$. In particular,

$$\Delta_1'(H) = -\Delta_1(H) = \frac{-V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{96 - 24}{288 - 192} = -0.75,$$

and

$$\Delta_1'(T) = -\Delta_1(T) = \frac{-V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = \frac{4 - 0}{192 - 128} = -0.0625.$$

Regardless of what the values of $\Delta_1'(H) = -\Delta_1(H)$ and $\Delta_1'(T) = -\Delta_1(T)$ are, at time 2 we have (after replacing $X_1' + X_1 = (1.1)V_0$),

$$X_2' + X_2 = (1.1)^2 V_0 = (1.1)^2 (48.97) = 59.25.$$

Solution (c): We have seen that

$$Z_n(\omega) = Z_n(\omega_1, \cdots, \omega_n) = \left( \frac{p}{\bar{p}} \right) ^{\#H(\omega_1, \cdots, \omega_n)} \left( \frac{q}{\bar{q}} \right) ^{\#T(\omega_1, \cdots, \omega_n)},$$

where $\#H(\omega_1, \cdots, \omega_n)$ is the number of $H$ in $\omega_1, \cdots, \omega_n$, and similarly for $\#T(\omega_1, \cdots, \omega_n)$. Since $\frac{p}{\bar{p}} = \frac{3}{2}$ and $\frac{q}{\bar{q}} = \frac{1}{2}$, we have

$$Z_0(\omega) = Z_0 = 1, \text{ for all } \omega,$$

$$Z_1(H\omega_2\omega_3) = Z_1(H) = \frac{3}{2}, \text{ for all } \omega_2, \omega_3$$

$$Z_1(T\omega_2\omega_3) = Z_1(T) = \frac{1}{2}, \text{ for all } \omega_2, \omega_3$$

$$Z_2(HH\omega_3) = Z_2(HH) = \frac{9}{4}, \text{ for all } \omega_3$$

$$Z_2(HT\omega_3) = Z_2(HT) = \frac{3}{4}, \text{ for all } \omega_3$$

$$Z_2(TH\omega_3) = Z_2(TH) = \frac{3}{4}, \text{ for all } \omega_3$$

$$Z_2(TT\omega_3) = Z_2(TT) = \frac{1}{4}, \text{ for all } \omega_3.$$

(2) Consider the $N$-period Binomial model with risk neutral probability measure $\bar{P}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

$$Y_0 = 1, \text{ and } Y_n = \frac{1}{(1 + X_0) \cdots (1 + X_{n-1})}, \text{ for } n = 1, \cdots, N.$$

(a) Let $U_n = \bar{E}_n \left[ \frac{Y_N}{Y_n} \right], \text{ for } n = 0, 1, \cdots, N$. Show that the process $Y_0 U_0, Y_1 U_1, \cdots, Y_N U_N$ is a martingale with respect to $\bar{P}$. (1.5 pts)

(b) (Corrected version) Let $U_n$ be as given in part (a). Set $I_0 = 0$ and define $I_n = \sum_{j=0}^{n-1} Y_{j+1}(Y_{j+1} U_{j+1} - Y_j U_j), \text{ for } n = 1, \cdots, N$. Show that $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\bar{P}$. (1.5 pts)

Solution (a): First note that the process $\{X_n : n = 0, \cdots, N\}$ is adapted, hence the random variable $Y_n$ is known at time $n - 1$, i.e. depends on the first $n - 1$ tosses, $n = 1, \cdots, N$. Hence,

$$U_n = \bar{E}_n \left[ \frac{Y_N}{Y_n} \right] = \frac{1}{Y_n} \bar{E}_n [Y_N],$$

which implies $Y_n U_n = \bar{E}_n [Y_N]$. Using the iteration property of conditional expectations, or directly Theorem 3.2.1, one has that the process $Y_0 U_0, Y_1 U_1, \cdots, Y_N U_N$ is a martingale with respect to $\bar{P}$.
Solution (b): First note that $Y_{n+1}$ is known at time $n$, and
\[ I_{n+1} = I_n + Y_{n+1}(Y_{n+1}U_{n+1} - Y_nU_n). \]
From part (a), we have that $Y_0U_0, \cdots, Y_NU_N$ is a martingale with respect to $\mathbb{P}$, and hence $E_n(Y_{n+1}U_{n+1} - Y_nU_n) = 0$. Thus,
\[ \mathbb{E}_n(I_{n+1}) = I_n + Y_{n+1} \mathbb{E}_n(Y_{n+1}U_{n+1} - Y_nU_n) = I_n. \]
Therefore, $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\mathbb{P}$.

(3) Consider the $N$-period binomial model, with expiration process $N$, up factor $u$, down factor $d$ and interest rate $r$. Let $\mathbb{P}$ be the risk neutral probability and $\mathbb{P}$ the real probability. We denote by $p = \mathbb{P}(H)$ and $\tilde{p} = \mathbb{P}(H)$. Let $S_0, S_1, \cdots, S_N$ be the corresponding price process.

(a) Define $Y_n = \sum_{k=0}^n S_k$. Show that the process
\[ (Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N) \]
is Markov with respect to $\mathbb{P}$ and $\tilde{\mathbb{P}}$. (Hint: use the random variables $Z_{n+1} = \frac{S_{n+1}}{S_n}$ and the Independence Lemma). (2 pts)

(b) Let $V_N = \left( S_N - \frac{Y_N}{N+1} \right)^+$. Show that for each $n = 0, 1, \cdots, N$, there exists a function $f_n$ such that
\[ E_n(ZV_N) = Z_n(1 + r)^{N-n} f_n(Y_n, S_n), \]
where $Z$ is the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and $Z_n = E_n(Z)$, $n = 0, 1, \cdots, N$. (1 pt)

Solution (a): Define $Z_{n+1} = \frac{S_{n+1}}{S_n}$ for $n = 0, 1, \cdots, N - 1$. Then,
\[ Y_{n+1} = Y_n + Z_{n+1}S_n, \quad \text{and} \quad S_{n+1} = Z_{n+1}S_n. \]
Note that $Z_{n+1}$ is independent of the first $n$ tosses, while $Y_n$ and $S_n$ depend only on the first $n$ tosses. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be any function, by the Independence Lemma, we have
\[ E_n(f(Y_{n+1}, S_{n+1})) = E_n(f(Y_n + Z_{n+1}S_n, Z_{n+1}S_n)) = g(Y_n, S_n), \]
where
\[ g(y, s) = \mathbb{E}(f(y + Z_{n+1}s, Z_{n+1}s)) = pf(y + us, us) + qf(y + ds, ds). \]
A similar calculation shows that
\[ \mathbb{E}_n(f(Y_{n+1}, S_{n+1})) = \mathbb{E}_n(f(Y_n + Z_{n+1}S_n, Z_{n+1}S_n)) = h(Y_n, S_n), \]
where
\[ h(y, s) = \mathbb{E}(f(y + Z_{n+1}s, Z_{n+1}s)) = \tilde{p}f(y + us, us) + \tilde{q}f(y + ds, ds). \]
Hence, the process
\[ (Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N) \]
is Markov with respect to $\mathbb{P}$ and $\tilde{\mathbb{P}}$.

Solution (b): Let $f(y, s) = (s - y(n+1)^{-1})^+$, then $V_N = f(Y_N, S_N)$. Since $(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$ is Markov with respect to $\tilde{\mathbb{P}}$, by Theorem 2.5.8, for each $n = 0, 1, \cdots, N$, there exists a function $f_n$ such that
\[ V_n = \mathbb{E}_n(V_N(1 + r)^{-(N-n)}) = f_n(Y_n, S_n), \]
(note that $f = f_N$). Thus, by Lemma 3.2.6
\[ E_n(ZV_N) = Z_n\mathbb{E}_n(V_N) = Z_n(1 + r)^{N-n} f_n(Y_n, S_n). \]