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Solutions Mid-Term: Inleiding Financiele Wiskunde 2018-2019

(1) Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = 1$, $n = 1, 2, \cdots$. Consider the filtration $\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \cdots$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(n) = \sigma(X_1, \cdots, X_n)$, $n = 1, 2, \cdots$. For $n = 1, 2, \cdots$, let $S_n = \sum_{i=1}^{n} X_i$ and $M_n = S_n^2 - n$. Set $M_0 = 0$.

(a) Prove that the stochastic process $\{M_n : n = 0, 1, \cdots\}$ is adapted to the filtration $\{\mathcal{F}_n : n = 0, 1, \cdots\}$. (0.5 pts)

(b) Prove that $\{M_n : n = 0, 1, \cdots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n : n = 0, 1, \cdots\}$. (2 pts)

(c) Determine the value of $\mathbb{E}[M_n]$ for $n = 0, 1, \cdots$. (0.5 pt)

Proof(a): This follows from the fact that S_n is $\mathcal{F}(n)$-measurable and hence so is $M_n = S_n^2 - n$.

Proof(b): First note that $S_{n+1} = S_n + X_{n+1}$, and $S_{n+1}^2 = S_n^2 + 2S_nX_{n+1} + X_{n+1}^2$. Furthermore, S_n is $\mathcal{F}(n)$-measurable and X_{n+1} is independent of $\mathcal{F}(n)$. Thus,

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}(n)] &= \mathbb{E}[S_{n+1}^2 - (n+1)]|\mathcal{F}(n)] \\
&= \mathbb{E}[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|\mathcal{F}(n)] - (n+1) \\
&= S_n^2 + 2S_n\mathbb{E}[X_{n+1}] + \mathbb{E}[X_{n+1}^2] - (n+1) \\
&= S_n^2 + 1 - (n+1) \\
&= M_n.
\end{aligned}$$

This shows that $(M_n : n \geq 0)$ is a martingale.

Proof(c): Since $(M_n : n \geq 0)$ is a martingale, we have $\mathbb{E}[M_n] = \mathbb{E}[M_0] = 0$ for all $n \geq 0$.

(2) In Homework 1, you have seen that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{B_1, \cdots, B_n\}$ a finite **partition** of Ω with $B_i \in \mathcal{F}$ for $i = 1, 2, \cdots, n$ and $\mathcal{G} = \sigma(B_1, \cdots, B_n)$ the σ-algebra generated by the partition $\{B_1, \cdots, B_n\}$, then for any random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ one has

$$\mathbb{E}[X|\mathcal{G}] = \sum_{i=1}^{n} \frac{1}{\mathbb{P}(B_i)} \mathbb{I}_{B_i} \mathbb{E}[\mathbb{I}_{B_i} X].$$

Use this formula to show that if $X = \sum_{i=1}^{n} x_i \mathbb{I}_{\{X=x_i\}}$ and $Y = \sum_{j=1}^{m} y_i \mathbb{I}_{\{Y=y_j\}}$ are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$ respectively, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\sigma(Y)] = \sum_{j=1}^{m} \mathbb{I}_{\{Y=y_j\}} \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i|Y = y_j],$$

where

$$\mathbb{P}(X = x_i|Y = y_j) = \frac{\mathbb{P}(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{\mathbb{P}(\{X = x_i\} \cap \{Y = y_j\})}{P(\{Y = y_j\})}.$$

(2 pts)

Proof: Note that $\{\{Y = y_j\} : j = 1, \cdots, m\}$ is a partition of Ω with $\{Y = y_j\} \in \mathcal{F}$ for all $j = 1, \cdots, m$, and $\sigma(Y) = \sigma(\{Y = y_j\} : j = 1, \cdots, m)$. Now, for any $j = 1, \cdots, m$

$$\mathbb{I}_{\{Y=y_j\}} X = \mathbb{I}_{\{Y=y_j\}} \sum_{i=1}^{n} x_i \mathbb{I}_{\{X=x_i\}} = \sum_{i=1}^{n} x_i \mathbb{I}_{\{X=x_i, Y=y_j\}}.$$

Hence,

$$\mathbb{E}[\mathbb{I}_{\{Y=y_j\}} X] = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i, Y = y_j),$$

and

$$\frac{1}{\mathbb{P}(Y = y_j)} E[\mathbb{I}_{\{Y=y_j\}} X] = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i | Y = y_j).$$

Therefore,

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\sigma(Y)] = \sum_{j=1}^{m} \mathbb{I}_{\{Y=y_j\}} \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i | Y = y_j].$$

In other words, if $\omega \in \Omega$ satsifies $Y(\omega) = y_j$, then

$$E[X|Y](\omega) := \mathbb{E}[X|Y = y_j] = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i | Y = y_j].$$

(3) Let $(W(t) : t \geq 0)$ be a Brownian motion, and let $\{\mathcal{F}_t : t \geq 0\}$ be its natural filtration, i.e. $\mathcal{F}(t) = \sigma(W(s) : s \leq t)$. Consider the stochastic process $(X(t) : t \geq 0)$ defined by $X(t) = e^{\mu c(t) + W(t)}$ with $\mu \neq 0$, and $c(t)$ a (measurable) function satisfying $c(0) = \alpha$, with $\alpha \neq 0$ some given real number. Suppose we are told that the process $(X(t) : t \geq 0)$ is a martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

(a) Determine an explicit expression for $c(t)$. (1 pt)

(b) Determine the value of $\mathbb{P}(X(t) > e^{\mu\alpha})$, for $t > 0$. (1 pt)

(c) Let a be a real number. Prove that $\mathbb{E}[W(t)\, e^{aW(t)}] = at\, e^{\frac{1}{2}a^2 t}$ and $\mathbb{E}[W(t)^2\, e^{aW(t)}] = (t + a^2 t^2)\, e^{\frac{1}{2}a^2 t}$. (1 pt)

(d) Let a be a real number and $0 \leq s < t$. Use part (c) to prove that

$$\mathbb{E}[W(t) e^{aW(t)} | \mathcal{F}(s)] = [W(s) + a(t - s)] e^{\frac{1}{2}a^2(t-s) + aW(s)}.$$

(2 pts)

Proof (a): Since $(X(t) : t \geq 0)$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$, we have $\mathbb{E}[X(t)] = \mathbb{E}[X_0] = e^{\mu\alpha}$, for any $t \geq 0$. Now,

$$\mathbb{E}[X(t)] = \mathbb{E}[e^{\mu c(t) + W(t)}] = e^{\mu c(t)} \mathbb{E}[e^{W(t)}] = e^{\mu c(t)} e^{\frac{1}{2}t} = e^{\mu c(t) + \frac{1}{2}t},$$

where in the last step we used the fact that the moment generating function of a normal random variable Y with mean 0 and variance t is given by $E[e^{aY}] = e^{\frac{1}{2}a^2 t}$. Since $\mathbb{E}[X(t)] = \mathbb{E}[X_0] = e^{\mu\alpha}$, we then have $\mu c(t) + \frac{1}{2}t = \mu\alpha$ or equivalently, $c(t) = \alpha - \dfrac{t}{2\mu}$.

Proof (b): Recall that $W(t)$ is normally distributed with mean zero and variance t, hence $W'(t) = W(t)/\sqrt{t}$ has the standard normal distribution. We denote (as in the book) the distribution of the standard normal with N. From part (a), we have $X(t) = e^{\mu\alpha - t/2 + W(t)}$, thus

$$\begin{aligned}
\mathbb{P}(X(t) > e^{\mu\alpha)} &= \mathbb{P}(e^{\mu\alpha - t/2 + W(t)} > e^{\mu\alpha}) \\
&= \mathbb{P}(\mu\alpha - t/2 + W(t) > \mu\alpha) \\
&= \mathbb{P}(W(t) > t/2) \\
&= \mathbb{P}(W'(t) > \sqrt{t}/2) \\
&= 1 - N(\sqrt{t}/2).
\end{aligned}$$

Proof (c): Recall that $W(t)$ is $\mathcal{N}(0,t)$ distributed. For $k \in \{1,2\}$ we have

$$\mathbb{E}[W^k(t)\, e^{aW(t)}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} w^k e^{aw} e^{-\frac{1}{2}\frac{w^2}{t}}\, dw = e^{\frac{1}{2}a^2 t} \int_{-\infty}^{\infty} \frac{w^k}{\sqrt{2\pi t}} e^{-\frac{1}{2}(\frac{w-at}{\sqrt{t}})^2}\, dw.$$

Note that $\int_{-\infty}^{\infty} \frac{w^k}{\sqrt{2\pi t}} e^{-\frac{1}{2}(\frac{w-at}{\sqrt{t}})^2}\, dw = \mathbb{E}(Z^k)$, where Z is normally distributed with mean at and variance t, hence it has value at if $k=1$ and value $t + a^2t^2$ if $k=2$. This shows that $\mathbb{E}[W(t)\, e^{aW(t)}] = at\, e^{\frac{1}{2}a^2 t}$ and $\mathbb{E}[W(t)^2\, e^{aW(t)}] = (t + a^2t^2)\, e^{\frac{1}{2}a^2 t}$.

Proof (d): Using linearity, we have

$$\begin{aligned}\mathbb{E}[W(t)e^{aW(t)}|\mathcal{F}(s)] &= \mathbb{E}[(W(t)-W(s))e^{a(W(t)-W(s))}\, e^{aW(s)}|\mathcal{F}(s)] \\ &\quad + \mathbb{E}[W(s)\, e^{a(W(t)-W(s))}\, e^{aW(s)}|\mathcal{F}(s)].\end{aligned}$$

Since $W(t)-W(s)$ is independent of $\mathcal{F}(s)$, while $W(s)$ is $\mathcal{F}(s)$-measurable, we have

$$\begin{aligned}\mathbb{E}[W(t)\, e^{aW(t)}|\mathcal{F}(s)] &= e^{aW(s)}\, \mathbb{E}[(W(t)-W(s))\, e^{a(W(t)-W(s))}] \\ &\quad + W(s)\, e^{aW(s)}\, \mathbb{E}[e^{a(W(t)-W(s))}].\end{aligned}$$

By part (c) we have

$$\begin{aligned}\mathbb{E}[W(t)\, e^{aW(t)}|\mathcal{F}(s)] &= e^{aW(s)}\, a(t-s)e^{\frac{1}{2}a^2(t-s)} + W(s)\, e^{aW(s)}\, e^{\frac{1}{2}a^2(t-s)}] \\ &= [W(s) + a(t-s)]\, e^{aW(s)+\frac{1}{2}a^2(t-s)}.\end{aligned}$$