Deeltentamen 1: Inleiding Finaniele Wiskunde 2017-2018

(1) Consider an 2-period binomial model with $S_0 = 200$, $u = 1.2$, $d = 0.8$, and $r = 0.1$. Suppose the real probability measure $\mathbb{P}$ satisfies $\mathbb{P}(H) = p = \dfrac{1}{2} = \mathbb{P}(T)$. Denote the risk-neutral probability (as usual) by $\widetilde{\mathbb{P}}$.

(a) Consider an option with payoff $V_2 = \left(\frac{1}{4}S_1 + \frac{3}{4}S_2 - 180\right)^+$. Determine the price V_n at time $n = 0, 1$. (1 pt)

(b) Suppose an investor owns the option and intends to hold it until the expiration date ($N = 2$) and receives the payoff V_2. So, at time 0 the investor has a capital of V_0 (your answer obtained in part (a)) which is tied up to the option and wants to earn the interest rate of 10% on this capital until time 2 without investing anymore money, and regardless of how the coin tossing turn out, the investor wants to have $(1.1)^2 V_0$. Determine a strategy (i.e. a wealth process) made of stocks and money market that accomplishes this goal. (2 pts)

(c) Consider the Radon-Nikodym derivative process Z_0, Z_1, Z_2, where $Z_n = \mathbb{E}_n(Z)$ and $Z(\omega) = Z(\omega_1\omega_2\omega_3) = \dfrac{\widetilde{\mathbb{P}}(\omega_1\omega_2\omega_3)}{\mathbb{P}(\omega_1\omega_2\omega_3)}$. Determine explicitly the values of $Z_n(\omega)$ for $n = 0, 1, 2$ and all $\omega = (\omega_1, \omega_2, \omega_3)$. (1 pt)

(2) Consider the N-period Binomial model with risk neutral probability measure $\widetilde{\mathbb{P}}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

$$Y_0 = 1, \quad \text{and } Y_n = \frac{1}{(1 + X_0)\cdots(1 + X_{n-1})}, \; n = 1, \cdots, N.$$

(a) Let $U_n = \widetilde{\mathbb{E}}_n\left[\dfrac{Y_N}{Y_n}\right]$, $n = 0, 1, \cdots, N$. Show that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$. (1.5 pts)

(b) Let U_n be as given in part (a). Set $I_0 = 0$ and define $I_n = \displaystyle\sum_{j=0}^{n-1} Y_{j+1}(Y_{j+1}U_{j+1} - Y_jU_j)$, $n = 1, \cdots, N$. Show that $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{\mathbb{P}}$. (1.5 pts)

(3) Consider the N-period binomial model, with expiration process N, up factor u, down factor d and interst rate r. Let $\widetilde{\mathbb{P}}$ be the risk neutral probability and $\mathbb{P}$ the real probability. We denote by $p = \mathbb{P}(H)$ and $\widetilde{p} = \widetilde{\mathbb{P}}(H)$. Let $S_0, S_1, \cdots, S_N$ be the corresponding price process.

(a) Define $Y_n = \sum_{k=0}^{n} S_k$. Show that the process

$$(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$$

is Markov with respect to $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. (Hint: use the random variables $Z_{n+1} = \dfrac{S_{n+1}}{S_n}$ and the Independence Lemma). (2 pts)

(b) Let $V_N = \left(S_N - \dfrac{Y_N}{N+1}\right)^+$. Show that for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$E_n(ZV_N) = Z_n(1 + r)^{N-n} f_n(Y_n, S_n),$$

where Z is the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and $Z_n = \mathbb{E}_n(Z)$, $n = 0, 1, \cdots, N$. (1 pt)