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Mid-Term: Inleiding Financiele Wiskunde 2018-2019

(1) Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = 1$, $n = 1, 2, \cdots$. Consider the filtration $\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \cdots$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(n) = \sigma(X_1, \cdots, X_n)$, $n = 1, 2, \cdots$. For $n = 1, 2, \cdots$, let $S_n = \sum_{i=1}^{n} X_i$ and $M_n = S_n^2 - n$. Set $M_0 = 0$.

(a) Prove that the stochastic process $\{M_n : n = 0, 1, \cdots\}$ is adapted to the filtration $\{\mathcal{F}_n : n = 0, 1, \cdots\}$. (0.5 pts)

(b) Prove that $\{M_n : n = 0, 1, \cdots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n : n = 0, 1, \cdots\}$. (2 pts)

(c) Determine the value of $\mathbb{E}[M_n]$ for $n = 0, 1, \cdots$. (0.5 pt)

(2) In Homework 1, you have seen that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{B_1, \cdots, B_n\}$ a finite **partition** of Ω with $B_i \in \mathcal{F}$ for $i = 1, 2, \cdots, n$ and $\mathcal{G} = \sigma(B_1, \cdots, B_n)$ the σ-algebra generated by the partition $\{B_1, \cdots, B_n\}$, then for any random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ one has

$$\mathbb{E}[X|\mathcal{G}] = \sum_{i=1}^{n} \frac{1}{\mathbb{P}(B_i)} \mathbb{I}_{B_i} \mathbb{E}[\mathbb{I}_{B_i} X].$$

Use this formula to show that if $X = \sum_{i=1}^{n} x_i \mathbb{I}_{\{X = x_i\}}$ and $Y = \sum_{j=1}^{m} y_i \mathbb{I}_{\{Y = y_j\}}$ are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$ respectively, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\sigma(Y)] = \sum_{j=1}^{m} \mathbb{I}_{\{Y = y_j\}} \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i | Y = y_j),$$

where

$$\mathbb{P}(X = x_i | Y = y_j) = \frac{\mathbb{P}(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{\mathbb{P}(\{X = x_i\} \cap \{Y = y_j\})}{P(\{Y = y_j\})}.$$

(2 pts)

(3) Let $(W(t) : t \geq 0)$ be a Brownian motion, and let $\{\mathcal{F}_t : t \geq 0\}$ be its natural filtration, i.e. $\mathcal{F}(t) = \sigma(W(s) : s \leq t)$. Consider the stochastic process $(X(t) : t \geq 0)$ defined by $X(t) = e^{\mu c(t) + W(t)}$ with $\mu \neq 0$, and $c(t)$ a (measurable) function satisfying $c(0) = \alpha$, with $\alpha \neq 0$ some given real number. Suppose we are told that the process $(X(t) : t \geq 0)$ is a martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

(a) Determine an explicit expression for $c(t)$. (1 pt)

(b) Determine the value of $\mathbb{P}(X(t) > e^{\mu\alpha})$, for $t > 0$. (1 pt)

(c) Let a be a real number. Prove that $\mathbb{E}[W(t)\, e^{aW(t)}] = at\, e^{\frac{1}{2}a^2 t}$ and $\mathbb{E}[W(t)^2\, e^{aW(t)}] = (t + a^2 t^2)\, e^{\frac{1}{2}a^2 t}$. (1 pt)

(d) Let a be a real number and $0 \leq s < t$. Use part (c) to prove that

$$\mathbb{E}[W(t) e^{aW(t)} | \mathcal{F}(s)] = [W(s) + a(t - s)] e^{\frac{1}{2}a^2(t-s) + aW(s)}.$$

(2 pts)