Mathematical Modeling

WISB357

Universiteit Utrecht
Blok 2, 2018/19
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Final Exam

Name:

Student number:

Signature:

Date: Monday, January 28, 2019

Time: 13:30 - 16:30 (3 hours)

Room: BBG 023

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, notes, calculators, cell phones, etc. is *not* allowed.
- Make sure that your answers are *readable* and *understandable*.
- Problems marked with * are bonus questions.

Total points: 60 (including bonus points)

Score:

1	2	3	4	Σ

Grade:

Exercise 1 (Spreading of diseases)

The evolution of chronic infectious diseases like HIV or hepatitis C within a population of humans can be modeled by the system of ordinary differential equations

$$\begin{cases} S' &= \Lambda - \beta IS - \mu S, \\ I' &= \beta IS - \eta I, \end{cases} \tag{1}$$

where $S, I : [0, \infty) \to [0, \infty)$ and $\Lambda, \beta, \mu, \eta > 0$.

a) Give an interpretation of all the quantities appearing in (1). What is a plausible assumption on the relation between μ and η? Explain your answer. 3p

b) Non-dimensionalize (1) to obtain a system of the form

$$\begin{cases} x' &= \rho(1 - x) - \mathcal{R}xy, \\ y' &= (\mathcal{R}x - 1)y. \end{cases} \tag{2}$$

Express the parameters $\rho > 0$ and $\mathcal{R} > 0$ in terms of $\beta, \Lambda, \mu, \eta$. 5p

Henceforth, we assume that $\mathcal{R} > 1$.

c) Find all stationary points of (2) and analyze their behavior regarding linearized stability. Are these stationary points (asymptotically) stable? 10p

d) Sketch a phase portrait for (2), including stationary points, isoclines and areas of monotonicity. 3p

Exercise 2 (Asymptotic expansion)

For a small parameter $\varepsilon > 0$, consider the initial value problem

$$y'' + 2\varepsilon y' + (1 + \varepsilon^2)y = 0, \quad y(0) = 0 \quad y'(0) = 1, \tag{3}$$

with unknown $y : [0, \infty) \to \mathbb{R}$.

a) Calculate an approximation for the solution to (3) by means of a formal asymptotic expansion up to first order.

Hint: The general solution to $z''(t) + z(t) = -2\cos t$ for $t \in \mathbb{R}$ is $z(t) = \alpha \sin t + \beta \cos t - t \sin t$ for $t \in \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$. 5p

b) Estimate the error between the exact solution to (3) and the first order approximation from a). Sketch the graphs of the two functions.

Hint: You may use that the explicit solution to (3) is $y(t) = e^{-\varepsilon t} \sin t$ for $t \geq 0$. 5p

Exercise 3 (Variational problem)

Consider the variational problem

$$\text{Minimize } \mathcal{I}(u) = \int_0^1 \frac{1}{2}u'(x)^2 - u(x)\,\mathrm{d}x \quad \text{for } u \in C^1([0,1]). \tag{4}$$

a) What is the (maximal) set of admissible perturbations of $u \in C^1([0,1])$? Determine the first variation $\delta\mathcal{I}(u)(\varphi)$ for all admissible perturbations φ. 5p

b) Show that

$$u'' = -1, \quad u'(0) = 0, \quad u'(1) = 0$$

are necessary conditions for a solution $u \in C^2([0,1])$ to (4). 8p

Exercise 4 (Linear elasticity)

The differential equations of linear elasticity for isotropic, homogeneous materials are

$$\rho\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = f \qquad \text{on } (0,\infty)\times\Omega. \tag{5}$$

Here, $\Omega \subset \mathbb{R}^3$ is the reference configuration of the body, u is the displacement field, and f a given force density per unit volume. Moreover, $\rho > 0$ is the constant mass density, and $\lambda \in \mathbb{R}$ and $\mu > 0$ are the Lamé constants.
Note that the differential operators ∇, $\nabla\cdot$ and Δ refer only to the spatial variable $x \in \Omega$.

a) Let $\Omega \subset \mathbb{R}^3$ be an open, bounded domain with smooth boundary. Assuming the initial and boundary conditions

$$\begin{cases} u = 0, \quad \partial_t u = 0 & \text{on } \{0\}\times\Omega, \\ \partial_t u = 0 & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{6}$$

derive the energy conservation identity

$$\int_\Omega \rho|\partial_t u(t,x)|^2 + \mu|\nabla u(t,x)|^2 + (\lambda+\mu)|\nabla\cdot u(t,x)|^2\,\mathrm{d}x = 2\int_0^t\int_\Omega f(\tau,x)\cdot\partial_t u(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$

for all $t > 0$. 9p

b) Let $\Omega = (0,1)^3$ and $f = 0$, and recall that e_1 denotes the first standard basis vector in $\mathbb{R}^3$. Consider the displacement field

$$u(t,x) = w(t,x_2)e_1 \qquad \text{for } x = (x_1,x_2,x_3) \in \Omega \text{ and } t > 0$$

with $w : (0,\infty)\times(0,1) \to \mathbb{R}$, which corresponds to a shear deformation in e_1-direction.
Show that if this displacement field is a solution to (5), then $w = w(t,y)$ has to solve

$$\rho\partial_t^2 w - \mu\partial_y^2 w = 0. \tag{7}$$

4p

c)* Is the linear second order partial differential equation in (7) elliptic, hyperbolic or parabolic? Justify your answer. 3p

Sketch of suggested solutions

Please email errors and/or suggestions to c.kreisbeck@uu.nl.

Exercise 1

a) The number of susceptible and infected persons at time $t \geq 0$ is denoted by $S(t)$ and $I(t)$, respectively. The growth of the population is governed by an increase rate Λ independent of its current size. It is assumed that all "new" individuals are susceptible. Whenever susceptible individuals meet infected ones, they also get infected with a probability $\beta > 0$. Moreover, μ and η are the mortality rates per capita for susceptible and infected individuals, respectively. It is reasonable to assume that $\mu \leq \eta$, since infection will not have a positive effect on a person's life expectancy. If $\mu < \eta$, this models an increased chance of death for infected individuals.

b) We introduce the characteristic scales $\bar{t}$, $\bar{S}$, and $\bar{I}$ and set $\tau = \frac{t}{\bar{t}}$, $x = \frac{S}{\bar{S}}$ and $y = \frac{I}{\bar{I}}$. Then,

$$\begin{cases} \frac{\bar{S}}{\bar{t}} x' = \Lambda - \beta \bar{I}\bar{S}xy - \mu \bar{S}x, \\ \frac{\bar{I}}{\bar{t}} y' = \beta \bar{I}\bar{S}xy - \eta \bar{I}y. \end{cases}$$

Dividing both equations by their coefficient of the highest order derivative leads to

$$\begin{cases} x' = \frac{\Lambda \bar{t}}{\bar{S}} - \beta \bar{I}\bar{t}xy - \mu \bar{t}x, \\ y' = \beta \bar{t}\bar{S}xy - \eta \bar{t}y. \end{cases}$$

To obtain $\eta\bar{t} = 1$, we take $\bar{t} = \frac{1}{\eta}$. Furthermore, the condition $\beta\bar{I}\bar{t} = \beta\bar{t}\bar{S}$ implies $\bar{S} = \bar{I}$. Finally, requiring $\frac{\Lambda\bar{t}}{\bar{S}} = \mu\bar{t}$ gives $\bar{S} = \frac{\Lambda}{\mu}$. This means that

$$\rho = \frac{\mu}{\eta} > 0 \qquad \text{and} \qquad \mathcal{R} = \frac{\Lambda\beta}{\mu\eta} > 0.$$

c) The isoclines of the system are

$$\{(x, y) \in \mathbb{R}^2 : y = 0\},\ \{(x, y) \in \mathbb{R}^2 : x = \tfrac{1}{\mathcal{R}}\} \text{ and } \{(x, y) \in \mathbb{R}^2 : y = \tfrac{\rho}{\mathcal{R}}(\tfrac{1}{x} - 1)\}.$$

Considering only non-negative values for x and y, two stationary points in the case $\mathcal{R} > 1$, namely $(1, 0)$ and $(\frac{1}{R}, \rho(1 - \frac{1}{\mathcal{R}}))$.

For the Jacobian of the function $f(x, y) = \begin{pmatrix} \rho(1-x) - \mathcal{R}xy \\ (\mathcal{R}x - 1)y \end{pmatrix}$, we obtain

$$\nabla f(x, y) = \begin{pmatrix} -\rho - \mathcal{R}y & -\mathcal{R}x \\ \mathcal{R}y & \mathcal{R}x - 1 \end{pmatrix}.$$

In particular,

$$A := \nabla f(1, 0) = \begin{pmatrix} -\rho & -\mathcal{R} \\ 0 & \mathcal{R} - 1 \end{pmatrix},$$

has the eigenvalues $-\rho < 0$ and $\mathcal{R} - 1 > 0$ (recall that $\mathcal{R} > 1$). This implies that the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

has 0 as an unstable stationary point. By definition, the stationary point $(0, 1)$ for (2) is then linearly unstable, and the principle of linearized stability states that $(0, 1)$ is unstable.

To study linearized stability of the stationary point $(\frac{1}{\mathcal{R}}, \rho(1 - \frac{1}{\mathcal{R}}))$, consider the matrix

$$B := J(\tfrac{1}{\mathcal{R}}, \rho(1 - \tfrac{1}{\mathcal{R}})) = \begin{pmatrix} -\rho\mathcal{R} & -1 \\ \rho(\mathcal{R} - 1) & 0 \end{pmatrix}.$$

The sign of the real parts of the eigenvalues λ_1 and λ_2 of B can be determined from the determinant and trace of B. Since

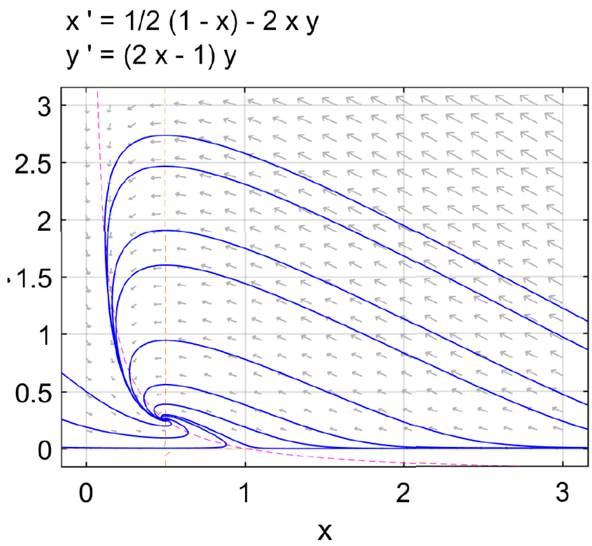
$$\lambda_1\lambda_2 = \det B = \rho(\mathcal{R} - 1) > 0 \qquad \text{and} \qquad \lambda_1 + \lambda_2 = \operatorname{tr} B = -\rho\mathcal{R} < 0,$$

both eigenvalues λ_1 and λ_2 have a negative real part. Thus, 0 is a asymptotically stable point for the linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

which means that $(\frac{1}{\mathcal{R}}, \rho(1-\frac{1}{\mathcal{R}}))$ is a linearly asymptotically stable stationary point for (2), provided $\mathcal{R} > 1$.

d) Considering the stationary points and isoclines determined in c), the phase portrait for (2) looks like this:



Exercise 2

a) To find the approximation via formal asymptotic expansion up to first order, one needs to determine the first two coefficient functions in

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots \tag{10}$$

where y is the solution to (3).

We plug the ansatz (10) into (3) and equate the terms of the same order in ε. The zeroth order term gives rise to the initial value problem

$$y_0'' + y_0 = 0, \quad y_0(0) = 0, \quad y_0'(0) = 1,$$

which has the unique solution $y_0(t) = \sin t$ for $t \geq 0$. Considering the first order terms implies that the second coefficient function y_1 has to satisfy

$$y_1'' + y_1 = -2y_0', \quad y_1(0) = 0, \quad y_1'(0) = 0. \tag{11}$$

Considering that $y_0'(t) = \cos t$ for $t > 0$ and the hint, the solution to (11) is $y_1(t) = -t \sin t$ for $t \geq 0$.

Finally, we obtain

$$y_{app}(t) := y_0(t) + \varepsilon y_1(t) = (1 - \varepsilon t) \sin t \qquad \text{for } t \geq 0,$$

as the first order approximation to the solution of (3).

b) By Taylor expansion of $t \mapsto e^{-\varepsilon t}$ at 0, we have for any $t > 0$ that

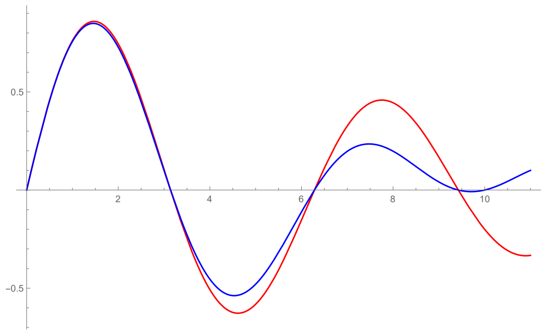
$$e^{-\varepsilon t} = 1 - \varepsilon t + \frac{1}{2}\varepsilon^2 e^{-\varepsilon \tau} t^2,$$

with some $\tau \in (0, t)$. This leads to the error estimate

$$|y(t) - y_{app}(t)| = |e^{-\varepsilon t} \sin t - (1 - \varepsilon t) \sin t| \leq |e^{-\varepsilon t} - 1 + \varepsilon t| \leq \frac{1}{2}\varepsilon^2 t^2$$

for all $t \geq 0$.

In the plot below, the blue and red graphs correspond to y and y_{app}, respectively.



Exercise 3

a) The maximal set of admissible perturbations of $u \in C^1([0,1])$ is $C^1([0,1])$. For $\varphi \in C^1([0,1])$, we obtain for the first variation of $\mathcal{I}$ that

$$\begin{aligned}
\delta I(u)(\varphi) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \int_0^1 \frac{1}{2}(u' + \varepsilon\varphi')^2 - (u + \varepsilon\varphi)\, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \int_0^1 \frac{1}{2}(u')^2 - u\, \mathrm{d}x + \varepsilon \int_0^1 u'\varphi' - \varphi\, \mathrm{d}x + \varepsilon^2 \int_0^1 (\varphi')^2\, \mathrm{d}x \\
&= \int_0^1 u'\varphi' - \varphi\, \mathrm{d}x.
\end{aligned}$$

b) If $u \in C^2([0,1])$ is a solution to (4), then u is a critical point of $\mathcal{I}$, i.e., $\delta\mathcal{I}(u)(\varphi) = 0$ for all $\varphi \in C^1([0,1])$. First, we take an arbitrary $\varphi \in C^1([0,1])$ with $\varphi(0) = \varphi(1) = 0$. Then, by a) and integration by parts,

$$0 = \delta\mathcal{I}(u)(\varphi) = \int_0^1 u'\varphi' - \varphi\, \mathrm{d}x = -\int_0^1 (u'' + 1)\varphi\, \mathrm{d}x.$$

Since this holds in particular for any $\varphi \in C_c^\infty(0,1)$, the fundamental theorem of the calculus of variations yields the pointwise identity $u'' + 1 = 0$, or

$$u'' = -1, \tag{12}$$

as stated.

To find that $u'(0) = 0$, take now any $\varphi \in C^1([0,1])$ with $\varphi(1) = 0$. Then, integration by parts and (12) lead to

$$0 = \delta\mathcal{I}(u)(\varphi) = \int_0^1 u'\varphi' - \varphi \,\mathrm{d}x = -\int_0^1 (u''+1)\varphi \,\mathrm{d}x + u'(0)\varphi(0) = u'(0)\varphi(0).$$

Finally, varying the boundary values $\varphi(0)$ implies that $u'(0) = 0$.

An analogous argument shows that $u'(1) = 0$.

Exercise 4

a) The idea is to (scalar) multiply (5) with $\partial_t u$ and to integrate over Ω, which gives

$$\int_\Omega \rho\partial_t^2 u \cdot \partial_t u - \mu\Delta u \cdot \partial_t u - (\lambda+\mu)\nabla(\nabla\cdot u)\cdot\partial_t u \,\mathrm{d}x = \int_\Omega f\cdot\partial_t u \,\mathrm{d}x.$$

With $\partial_t(|\partial_t u|^2) = \partial_t^2 u \cdot \partial_t u$, we can rewrite the first term as

$$\int_\Omega \rho\partial_t^2 u\cdot\partial_t u \,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho\partial_t\big(|\partial_t u|^2\big)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\partial_t u|^2\,\mathrm{d}x.$$

For the second term, it holds that

$$\begin{aligned} -\int_\Omega \mu\Delta u\cdot\partial_t u \,\mathrm{d}x &= \mu\int_\Omega \nabla u : \partial_t\nabla u \,\mathrm{d}x - \mu\int_{\partial\Omega}(\nabla u n)\cdot\partial_t u \,\mathrm{d}s_x \\ &= \int_\Omega \frac{\mu}{2}\partial_t|\nabla u|^2\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu|\nabla u|^2\,\mathrm{d}x, \end{aligned}$$

where, after integration by parts, we have used the identity $\frac{1}{2}\partial_t(|\nabla u|^2) = \nabla u : \partial_t\nabla u$ and exploited the boundary condition $\partial_t u = 0$ on $(0,\infty)\times\partial\Omega$. With the help of integration by parts, the third term becomes

$$\begin{aligned} -\int_\Omega(\lambda+\mu)\nabla(\nabla\cdot u)\cdot\partial_t u\,\mathrm{d}x &= (\lambda+\mu)\int_\Omega(\nabla\cdot u)\cdot\partial_t(\nabla\cdot u)\,\mathrm{d}x - (\lambda+\mu)\int_{\partial\Omega}(\nabla\cdot u)\partial_t u\cdot n\,\mathrm{d}s_x \\ &= (\lambda+\mu)\int_\Omega \frac{1}{2}\partial_t|\nabla\cdot u|^2\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\lambda+\mu)|\nabla\cdot u|^2\,\mathrm{d}x, \end{aligned}$$

where we have again exploited the boundary values $\partial_t u = 0$ on $(0,\infty)\times\partial\Omega$ and the identity $\frac{1}{2}\partial_t(|\nabla\cdot u|^2) = (\nabla\cdot u)\partial_t(\nabla\cdot u)$.

Finally, integrating from 0 to t for any $t > 0$ and using the fundamental theorem of calculus while accounting for the initial values leads to

$$\frac{1}{2}\int_\Omega \rho|\partial_t u(t,x)|^2 + \mu|\nabla u(t,x)|^2 + (\lambda+\mu)|\nabla\cdot u(t,x)|^2\,\mathrm{d}x = \int_0^t\int_\Omega f(\tau,x)\partial_t u(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

b) With the ansatz

$$u(t,x) = w(t,x_2)e_1 \qquad \text{for } x = (x_1,x_2,x_3)\in\Omega \text{ and } t>0,$$

we obtain that

$$\partial_t^2 u = \partial_t^2 w e_1, \quad \Delta u = \partial_y^2 w e_1, \quad \nabla \cdot u = 0.$$

Since u satisfies (5), we conclude for w that

$$\rho \partial_t^2 w(t,y) - \mu \partial_y^2 w(t,y) = 0 \text{ for } (t,y) \in (0,\infty) \times (0,1).$$

c) The coefficient matrix A for this second order linear PDE (7) is given by

$$A = \begin{pmatrix} \rho & 0 \\ 0 & -\mu \end{pmatrix},$$

which has the eigenvalues $\rho > 0$ and $-\mu < 0$. Hence, (7) a hyperbolic.