Mathematical Modeling

WISB357

Universiteit Utrecht
Blok 2, 2018/19
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Final Exam

Name:

Student number:

Signature:

Date: Monday, January 28, 2019

Time: 13:30 - 16:30 (3 hours)

Room: BBG 023

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, notes, calculators, cell phones, etc. is *not* allowed.
- Make sure that your answers are *readable* and *understandable*.
- Problems marked with * are bonus questions.

Total points: 60 (including bonus points)

Score:

1	2	3	4	Σ

Grade:

Exercise 1 (Spreading of diseases)

The evolution of chronic infectious diseases like HIV or hepatitis C within a population of humans can be modeled by the system of ordinary differential equations

$$\begin{cases} S' &= \Lambda - \beta IS - \mu S, \\ I' &= \beta IS - \eta I, \end{cases} \tag{1}$$

where $S, I : [0, \infty) \to [0, \infty)$ and $\Lambda, \beta, \mu, \eta > 0$.

a) Give an interpretation of all the quantities appearing in (1). What is a plausible assumption on the relation between μ and η? Explain your answer. 3p

b) Non-dimensionalize (1) to obtain a system of the form

$$\begin{cases} x' &= \rho(1-x) - \mathcal{R}xy, \\ y' &= (\mathcal{R}x - 1)y. \end{cases} \tag{2}$$

Express the parameters $\rho > 0$ and $\mathcal{R} > 0$ in terms of $\beta, \Lambda, \mu, \eta$. 5p

Henceforth, we assume that $\mathcal{R} > 1$.

c) Find all stationary points of (2) and analyze their behavior regarding linearized stability. Are these stationary points (asymptotically) stable? 10p

d) Sketch a phase portrait for (2), including stationary points, isoclines and areas of monotonicity. 3p

Exercise 2 (Asymptotic expansion)

For a small parameter $\varepsilon > 0$, consider the initial value problem

$$y'' + 2\varepsilon y' + (1 + \varepsilon^2)y = 0, \quad y(0) = 0 \quad y'(0) = 1, \tag{3}$$

with unknown $y : [0, \infty) \to \mathbb{R}$.

a) Calculate an approximation for the solution to (3) by means of a formal asymptotic expansion up to first order.

Hint: The general solution to $z''(t) + z(t) = -2\cos t$ for $t \in \mathbb{R}$ is $z(t) = \alpha \sin t + \beta \cos t - t \sin t$ for $t \in \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$. 5p

b) Estimate the error between the exact solution to (3) and the first order approximation from a). Sketch the graphs of the two functions.

Hint: You may use that the explicit solution to (3) is $y(t) = e^{-\varepsilon t} \sin t$ for $t \geq 0$. 5p

Exercise 3 (Variational problem)

Consider the variational problem

$$\text{Minimize } \mathcal{I}(u) = \int_0^1 \frac{1}{2}u'(x)^2 - u(x)\,\mathrm{d}x \qquad \text{for } u \in C^1([0,1]). \tag{4}$$

a) What is the (maximal) set of admissible perturbations of $u \in C^1([0,1])$? Determine the first variation $\delta\mathcal{I}(u)(\varphi)$ for all admissible perturbations φ. 5p

b) Show that

$$u'' = -1, \quad u'(0) = 0, \quad u'(1) = 0$$

are necessary conditions for a solution $u \in C^2([0,1])$ to (4). 8p

Exercise 4 (Linear elasticity)

The differential equations of linear elasticity for isotropic, homogeneous materials are

$$\rho\partial_t^2 u - \mu\Delta u - (\lambda + \mu)\nabla(\nabla \cdot u) = f \qquad \text{on } (0,\infty) \times \Omega. \tag{5}$$

Here, $\Omega \subset \mathbb{R}^3$ is the reference configuration of the body, u is the displacement field, and f a given force density per unit volume. Moreover, $\rho > 0$ is the constant mass density, and $\lambda \in \mathbb{R}$ and $\mu > 0$ are the Lamé constants.
Note that the differential operators ∇, $\nabla\cdot$ and Δ refer only to the spatial variable $x \in \Omega$.

a) Let $\Omega \subset \mathbb{R}^3$ be an open, bounded domain with smooth boundary. Assuming the initial and boundary conditions

$$\begin{cases} u = 0, \quad \partial_t u = 0 & \text{on } \{0\} \times \Omega, \\ \partial_t u = 0 & \text{on } (0,\infty) \times \partial\Omega, \end{cases} \tag{6}$$

derive the energy conservation identity

$$\int_\Omega \rho|\partial_t u(t,x)|^2 + \mu|\nabla u(t,x)|^2 + (\lambda+\mu)|\nabla \cdot u(t,x)|^2\,\mathrm{d}x = 2\int_0^t \int_\Omega f(\tau,x)\cdot\partial_t u(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$

for all $t > 0$. 9p

b) Let $\Omega = (0,1)^3$ and $f = 0$, and recall that e_1 denotes the first standard basis vector in $\mathbb{R}^3$. Consider the displacement field

$$u(t,x) = w(t,x_2)e_1 \qquad \text{for } x = (x_1,x_2,x_3) \in \Omega \text{ and } t > 0$$

with $w : (0,\infty) \times (0,1) \to \mathbb{R}$, which corresponds to a shear deformation in e_1-direction.
Show that if this displacement field is a solution to (5), then $w = w(t,y)$ has to solve

$$\rho\partial_t^2 w - \mu\partial_y^2 w = 0. \tag{7}$$

4p

c)* Is the linear second order partial differential equation in (7) elliptic, hyperbolic or parabolic? Justify your answer. 3p