Topologie en meetkunde – Retake exam

- **On solutions written on exam paper will be counted.**
- You can give solutions in English or Dutch.
- You are expected to explain your answers.
- You are allowed to use results of the lectures, the exercises and homework.
- All maps in the statements of the problems are meant to be continuous.
- Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones. It is deliberate that there are a lot of problems, probably more than you can solve; this is to the purpose that you can choose the problems which are easiest for you.

Problem 1 (8 points). *Give examples of non-contractible spaces X such that $\pi_1(X, x)$ is for every $x \in X$*

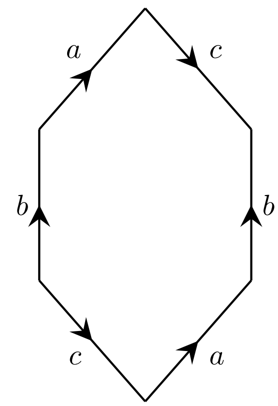
- *the group with only one element,*
- *isomorphic to $\mathbb{Z} \times \mathbb{Z}$,*
- *a non-abelian group.*

Problem 2 (8 points). *Show that the composition of two homotopy equivalences is a homotopy equivalence. You are allowed to use that the composition of two continuous maps is continuous and that being homotopic (for maps) is an equivalence relation, but apart from that, only argue from the definitions.*

Problem 3 (8 points). *Let $A \subset X$ be a path-connected subspace containing the base point $x_0 \in X$. Assume that $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. Let further $a, b \in A$. Show that every path from a to b in X is path homotopic to a path in A (with the same endpoints).*

Problem 4 (9 points). *Let $S^2 \subset \mathbb{R}^3$ be the 2-dimensional sphere. Let X be the union of S^2 with the line connecting $(1, 0, 0)$ with $(-1, 0, 0)$. Compute the fundamental group of X (with base point $(1, 0, 0)$).*

Problem 5 (9 points). *Define a surface S by glueing sides of a hexagon in the pattern depicted below. If $X_{m,n}$ is a surface obtained by attaching m cross-caps and n handles to a triangulated sphere, give all values of m and n such that S is homeomorphic to $X_{m,n}$.*



Problem 6 (8 points). *Let X and Y be two non-homeomorphic surfaces. Show for every $n \geq 2$ that $X \times S^n$ and $Y \times S^n$ are not homotopy equivalent.*

Problem 7 (20 points). *Let G be a graph. In this problem you are allowed to use that for every covering map $p\colon X \to G$ the space X has the structure of a graph again and p maps vertices to vertices and edges to edges (and the map from edge to edge is a homeomorphism on the interior). In the following we let G be the graph with one vertex and three edges so that $G \cong S^1 \vee S^1 \vee S^1$.*

(a) Give three examples of covering maps $X \to G$ such that the corresponding spaces X are pairwise non-homeomorphic. (You do not have to give proofs.)

(b) Let $p\colon X \to G$ be a covering map such that $p^{-1}(x)$ has for every $x \in G$ exactly two elements. Show that the Euler characteristic $\chi(X)$ equals -4. Moreover, assuming that X is connected, compute the fundamental group of X.

(c) Denote by F_n the free group on n letters, i.e. $\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ (with n copies of $\mathbb{Z}$). Let $H \subset F_3$ be a subgroup such that there is an element $g \in F_3$ so that every element of F_3 is either in H or in $gH = \{g \cdot h \mid h \in H\}$ (i.e. H has index 2 in F_3). Then $H \cong F_5$.