Retake Exam
(16th July 2018, 9:00 – 12:00)

• You are allowed to use one A4-sheet (recto-verso) of handwritten notes. By contrast, you are not allowed to use any book or electronic devices during the exam.

• Every question part is worth 1 point except where indicated otherwise. If you obtain 20 points, you are guaranteed a grade 10.

• Do not just give answers but prove your statements, e.g. by referring to a theorem in the course.

Question 1. Let $G$ be a finite group.

(a) For $H \trianglelefteq G$ a normal subgroup of order $|H| = 2$, show that $H \subseteq Z(G)$.

(b) Given a finite $G$-set $X$, define the associated permutation character $\chi_X$ and show that $\chi_X$ is irreducible if and only if $|X| = 1$.

(c) (2 points) Still for a finite $G$-set $X$, show that $\langle \chi_X, \chi_C \rangle = |X/G|$ is the number of orbits (where $\chi_C$ is the trivial $G$-character).

Question 2. Let $G$ be a finite group, $\psi: H \rightarrow \mathbb{C}^{\times}$ a linear character of a subgroup $H \leq G$ and $e_\psi := \frac{1}{|H|} \sum_{h \in H} \frac{h}{\psi(h)} \in \mathbb{C}H$.

(a) Show that $e_\psi$ is idempotent (i.e. $e_\psi^2 = e_\psi$).

(b) (2 points) Show that $\mathbb{C}e_\psi$ is a $\mathbb{C}H$-submodule of the regular module $\mathbb{C}H$ and that its character is $\psi$.

(c) (2 points) Show that $(\mathbb{C}e_\psi)^G \cong (\mathbb{C}G)e_\psi = \{xe_\psi | x \in \mathbb{C}G\} \leq \mathbb{C}G$ and conclude that the character of $(\mathbb{C}G)e_\psi$ is $\psi^G$.

Hint: You can even show that if $N \leq \mathbb{C}H$ is a $\mathbb{C}H$-submodule, then $N^G \cong (\mathbb{C}G)N$.

Question 3. Let $X$ be a $G$-set and consider $X^2 = X \times X$ as a $G$-set with the diagonal action (given by $g(x, x') = (gx, gx')$).

(a) Show that $\chi_{X^2} = (\chi_X)^2$ and conclude that $(\chi_X)^2 - 1$ is a well-defined character of $G$.

(b) Show that the action of $G$ is transitive (i.e. for all $x, y \in X$, there is $g \in G$ with $y = gx$) if and only if $\langle \chi_X - 1, \chi_C \rangle = 0$, where $\chi_C$ is the trivial $G$-character.

Now suppose that $G$ acts transitively and that $|X| \geq 2$. Such a transitive $G$-action is called doubly transitive iff for all $x \neq x', y \neq y' \in X$, we find a $g \in G$ such that $y = gx$ and $y' = gx'$.
(c) (2 points) Show that the following are equivalent:

(i) the $G$-action on $X$ is doubly transitive;
(ii) the $G$-action on $X^2$ has exactly two orbits;
(iii) $\langle (\chi_X)^2, \chi_C \rangle = 2$.

(d) Using characterisation (iii), conclude that $\chi_X - 1$ is irreducible iff the action of $G$ on $X$ is doubly transitive.

Question 4. Let $G = F_8$ be the so-called Frobenius group of order 56 (you don’t need to show this) given by

$$F_8 = \left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^7 = e, ab = ba, ac = ca, bc = cb, \right. \left. dad^{-1} = bc, dbd^{-1} = a, dcd^{-1} = b \right\rangle.$$ 

(a) For $\zeta := \exp(i2\pi/7)$, show that $\sum_{j=0}^6 \zeta^j = 0$. 

*Hint*: Geometric series

(b) Show that every element of $G$ is of the form $a^{i_1}b^{i_2}c^{i_3}d^j$ with $i_1, i_2, i_3 \in \{0, 1\}$ and $j \in \{0, \ldots, 6\}$ and that all $a^{i_1}b^{i_2}c^{i_3}$ form a conjugacy class.

(c) Determine the remaining conjugacy classes: For a fixed $j \in \{1, \ldots, 6\}$, show that the elements of the form $a^{i_1}b^{i_2}c^{i_3}d^j$ form a conjugacy class.

(d) Show that $N := \langle a, b, c \rangle \cong (C_2)^3 \triangleleft G$ is normal.

(e) Construct linear characters of $G$ by lifting from $G/N$.

(f) Complete the character table of $G$. 