RETAKE EXAM

(16^{th} July 2018, 9:00 – 12:00)

- You are allowed to use one A4-sheet (recto-verso) of handwritten notes. By contrast, you are not allowed to use any book or electronic devices during the exam.
- Every question part is worth 1 point except where indicated otherwise. If you obtain 20 points, you are guaranteed a grade 10.
- Do not just give answers but prove your statements, e.g. by referring to a theorem in the course.

Question 1. Let G be a finite group.

(a) For $H \trianglelefteq G$ a normal subgroup of order $|H| = 2$, show that $H \subseteq Z(G)$.

(b) Given a finite G-set X, define the associated permutation character χ_X and show that χ_X is irreducible if and only if $|X| = 1$.

(c) (2 points) Still for a finite G-set X, show that $\langle \chi_X, \chi_{\mathbb{C}} \rangle = |X/G|$ is the number of orbits (where $\chi_{\mathbb{C}}$ is the trivial G-character).

Question 2. Let G be a finite group, $\psi \colon H \to \mathbb{C}^\times$ a linear character of a subgroup $H \leqslant G$ and

$$e_\psi := \frac{1}{|H|} \sum_{h \in H} \frac{h}{\psi(h)} \in \mathbb{C}H.$$

(a) Show that e_ψ is idempotent (i.e. $e_\psi^2 = e_\psi$).

(b) (2 points) Show that $\mathbb{C}e_\psi$ is a $\mathbb{C}H$-submodule of the regular module $\mathbb{C}H$ and that its character is ψ.

(c) (2 points) Show that $(\mathbb{C}e_\psi)\!\uparrow^G \cong (\mathbb{C}G)e_\psi = \{xe_\psi \mid x \in \mathbb{C}G\} \leqslant \mathbb{C}G$ and conclude that the character of $(\mathbb{C}G)e_\psi$ is $\psi\!\uparrow^G$.
Hint: You can even show that if $N \leqslant \mathbb{C}H$ is a $\mathbb{C}H$-submodule, then $N\!\uparrow^G \cong (\mathbb{C}G)N$.

Question 3. Let X be a G-set and consider $X^2 = X \times X$ as a G-set with the diagonal action (given by $g(x, x') = (gx, gx')$).

(a) Show that $\chi_{X^2} = (\chi_X)^2$ and conclude that $(\chi_X)^2 - 1$ is a well-defined character of G.

(b) Show that the action of G is transitive (i.e. for all $x, y \in X$, there is $g \in G$ with $y = gx$) if and only if $\langle \chi_X - 1, \chi_{\mathbb{C}} \rangle = 0$, where $\chi_{\mathbb{C}}$ is the trivial G-character.

Now suppose that G acts transitively and that $|X| \geqslant 2$. Such a transitive G-action is called *doubly transitive* iff for all $x \neq x'$, $y \neq y' \in X$, we find a $g \in G$ such that $y = gx$ and $y' = gx'$.

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(c) (2 points) Show that the following are equivalent:

  (i) the G-action on X is doubly transitive;
  (ii) the G-action on X^2 has exactly two orbits;
  (iii) $\langle (\chi_X)^2, \chi_{\mathbb{C}} \rangle = 2$.

(d) Using characterisation (iii), conclude that $\chi_X - 1$ is irreducible iff the action of G on X is doubly transitive.

Question 4. Let $G = F_8$ be the so-called *Frobenius group* of order 56 (you don't need to show this) given by

$$F_8 = \left\langle a, b, c, d \;\middle|\; \begin{matrix} a^2 = b^2 = c^2 = d^7 = e,\ ab = ba,\ ac = ca,\ bc = cb, \\ dad^{-1} = bc,\ dbd^{-1} = a,\ dcd^{-1} = b \end{matrix} \right\rangle.$$

(a) For $\zeta := \exp(i2\pi/7)$, show that $\sum_{j=0}^{6} \zeta^j = 0$.
Hint: Geometric series

(b) Show that every element of G is of the form $a^{i_1} b^{i_2} c^{i_3} d^j$ with i_1, i_2, $i_3 \in \{0, 1\}$ and $j \in \{0, \ldots, 6\}$ and that all $a^{i_1} b^{i_2} c^{i_3}$ form a conjugacy class.

(c) Determine the remaining conjugacy classes: For a fixed $j \in \{1, \ldots, 6\}$, show that the elements of the form $a^{i_1} b^{i_2} c^{i_3} d^j$ form a conjugacy class.

(d) Show that $N := \langle a, b, c \rangle \cong (C_2)^3 \triangleleft G$ is normal.

(e) Construct linear characters of G by lifting from G/N.

(f) Complete the character table of G.