Question 1 (4 points)

a) Find the continued fraction expansion to $\sqrt{41}$.

b) What number has the continued fraction expansion $\langle 4, \overline{1,3,1,8} \rangle$?

Question 2 (4 points)

a) Find all integer solutions to the following system of congruences (i.e. integers $x$ that simultaneously solve all of the following congruences):

\[
\begin{align*}
x &\equiv 3 \pmod{6} \\
x &\equiv 6 \pmod{7} \\
x &\equiv 7 \pmod{143}.
\end{align*}
\]

b) Does the congruence

\[x^2 - 2x + 3 \equiv 0 \pmod{105}\]

have a solution?

Question 3 (4 points)

For a natural number $m$ let $\phi(m)$ be Euler’s phi-function, i.e. the number of invertible residue classes modulo $m$.

a) For what $n \in \mathbb{N}$ do we have $\phi(n) = 48$?

b) Compute the last digit of $3^{400}$.

Question 4 (4 points)

Show that

\[x \equiv a \pmod{m} \quad \text{and} \quad x \equiv b \pmod{n}\]

have a common solution if and only if $\gcd(m, n) \mid b - a$, and in this case the solution is unique modulo the least common multiple of $m$ and $n$. 

---

Date: 8th November 2018.
Question 5 (4 points)

Let $a, b, c, d \in \mathbb{Z}$ and $a \equiv d \equiv 4 \mod 9$. Assume that the equation

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = z^3$$

has a nontrivial integer solution in $x, y, z$ (i.e. a solution where not all of $x, y, z$ are equal to zero). Show that in this case it also has an integer solution with $3 \nmid xy$.

Question 6 (4 points)

Assume that the abc-conjecture holds. Show that there are only finitely many solutions $a, b, c, d, e, f \in \mathbb{N}$ to the equation

$$a^8b^9 + c^8d^9 = e^8f^9,$$

which satisfy $\gcd(ab, cd, ef) = 1$. Reminder: the natural numbers $\mathbb{N}$ do not contain 0 in the way that we defined it in the course.

Note: A simple non-programmable calculator is allowed for the exam.