F. Ziltener

blok 2, 2018/ 2019
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Functionaalanalyse, WISB315

Tentamen

Family name: ______________________ Given name: ______________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, **not** with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

(More instructions on the back.)

1	2	3	4	5	6	7	8	9	10	$\sum$
/5	/4	/6	/5	/4	/9	/7	/5	/7	/6	/58

You are **not** allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof:

- A given map is linear (if this is indeed the case).
- The sum of two continuous real-valued functions is continuous. Any multiple of a continuous real-valued function is continuous.
- Let $u, v : \mathbb{R} \to \mathbb{R}$ be differentiable functions, and $a \in \mathbb{R}$. Then $u + v$ is differentiable with derivative given by $u' + v'$. Furthermore, au is differentiable with derivative given by au'.
- A subset of a metric space is closed if and only if it is sequentially closed.
- A map between metric spaces is continuous if and only if it is sequentially continuous.
- The space ℓ^∞ of bounded sequences of numbers is not separable.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

23 points will suffice for a passing grade 6.

Good luck!

Problem 1 (norm on space of functions, 5 pt)**.** Let $k \in \mathbb{N} \cup \{0\}$. We define

$$X := \big\{ f : \mathbb{R} \to \mathbb{R} \,\big|\, f \text{ is } k \text{ times continuously differentiable, } f^{(i)} \text{ is bounded, } \forall i = 0, \dots, k \big\},$$
$$\| \cdot \| : X \to [0, \infty), \quad \|f\| := \max_{i=0,\dots,k} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|,$$

where $f^{(i)}$ denotes the i-th derivative of f. ($f^{(0)} = f$) Show the following:

(i) X is a linear subspace of the space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(ii) The map $\| \cdot \|$ is a norm.

Problem 2 (orthogonal complement, 4 pt)**.** Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space. We define the *orthogonal complement* of A to be the set

$$A^\perp := \big\{ x \in X \,\big|\, \langle x, y \rangle = 0, \, \forall y \in A \big\}.$$

Show that $A^\perp$ is a linear subspace of X and that it is closed in X.

Problem 3 (multiplication operator, 6 pt)**.** Let $p \in [1, \infty)$, and $x \in \ell^\infty = \ell^\infty(\mathbb{N})$. We define

$$M_x : \ell^p \to \ell^p, \quad M_x(y) := xy.$$

Show the following:

(i) The map M_x is well-defined, i.e., $M_x(y) \in \ell^p$ for every $y \in \ell^p$.

(ii) M_x is linear.

(iii) The operator norm of M_x is given by

$$\|M_x\| = \|x\|_\infty.$$

Problem 4 (ℓ^p separable, 5 pt)**.** Prove that for every $p \in [1, \infty)$ the space $\ell^p = \ell^p(\mathbb{N})$ is separable.

Problem 5 (prescribed Fourier coefficients, 4 pt)**.** We call a function $f : [0,1] \to \mathbb{R}$ Lebesgue-measurable iff the set $f^{-1}((-\infty, b])$ is Lebesgue-measurable for every $b \in \mathbb{R}$. Does there exist a Lebesgue-measurable function $f : [0,1] \to \mathbb{R}$, such that

$$\int_{[0,1]} |f|^2 d\lambda < \infty, \quad \widehat{f^n} = \frac{1}{\sqrt[4]{n}}, \, \forall n \in \mathbb{N}?$$

Remark: Here λ denotes the Lebesgue-measure on $[0,1]$ and $\widehat{f^n}$ the n-th Fourier coefficient of f.

(More problems on the back.)

Problem 6 (completeness of the quotient seminorm, 9 pt)**.** Let X be a real vector space and $Y \subseteq X$ a linear subspace. We denote by X/Y the quotient of X by Y. Let $\|\cdot\|$ be a seminorm on X. We define the quotient seminorm to be the map

$$\|\cdot\|^Y : X/Y \to [0,\infty), \quad \|\overline{x}\|^Y := \inf_{x \in \overline{x}} \|x\|.$$

Show that if $\|\cdot\|$ is complete then $\|\cdot\|^Y$ is complete.

Remark: You do *not* need to show that $\|\cdot\|^Y$ is indeed a seminorm.

Problem 7 (Hilbert space reflexive, 7 pt)**.** Prove that every Hilbert space H is reflexive.

Hint: Relate the canonical map $\iota_H : H \to H''$ to the map

$$\Phi_H : H \to H', \quad \Phi_H(y) := \langle \cdot, y \rangle.$$

Problem 8 (dual space of ℓ^∞, 5 pt)**.** Prove that the map

$$\Phi_\infty : \ell^1 \to (\ell^\infty)', \quad (\Phi_\infty y)x := \sum_{i=1}^{\infty} x^i y^i,$$

is not surjective.

Remark: You do *not* need to show that this map is well-defined.

Hint: Use another problem of this exam.

Problem 9 (spectrum of integral operator, 7 pt)**.** Let

$$T : X := C\big([0,1],\mathbb{C}\big) \to X, \quad (Tx)(t) := \int_0^t x(s)ds.$$

Show that the spectrum of T equals $\{0\}$.

Remark: The map T is well-defined and linear. You do *not* need to show this.

Hint: Compute the spectral radius of T.

Problem 10 (continuous bilinear map, 6 pt)**.** Let $(X_i, \|\cdot\|_i)$, $i = 0,1$, and $(Y, \|\cdot\|)$ be normed real vector spaces, and $b : X_0 \times X_1 \to Y$ a continuous bilinear map. Assume that X_1 is complete. Show that

$$\sup\Big\{\big\|b(x_0,x_1)\big\| \,\Big|\, x_i \in \overline{B}_1^{X_i}(0),\ \forall i = 0,1\Big\} < \infty.$$