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blok 2, 2018/ 2019
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# Functionaalanalyse, WISB315

## Hertentamen

Family name: ____________________ Given name: ____________________

Student number: ________________________________________

**Please:**

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, **not** with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are **not** allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result $X$ in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of $X$ without proving it.

Unless otherwise stated, you may use without proof:

- A given map is linear (if this is indeed the case).
- $\big(C([0,1],\mathbb{R}), \|\cdot\|_\infty\big)$ is a Banach space.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

23 points will suffice for a passing grade 6.

Good luck!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\sum$ |
|---|---|---|---|---|---|---|---|---|---|
| /7 | /8 | /6 | /7 | /6 | /5 | /4 | /6 | /6 | /55 |

**Problem 1** ($\ell^1$, $\ell^\infty$ are normed spaces, 7 pt)**.** Let $\mathbb{K} = \mathbb{R}$. For $p = 1$ and $p = \infty$ show that the set $\ell^p = \ell^p(\mathbb{N}, \mathbb{R})$ is a linear subspace of $\mathbb{R}^{\mathbb{N}}$ and that $\|\cdot\|_p$ is a norm on $\ell^p$.

**Remark:** Recall that

$$\|x\|_1 := \sum_{i\in\mathbb{N}} |x^i|, \quad \|x\|_\infty := \sup_{i\in\mathbb{N}} |x^i|.$$

**Problem 2** (bounded linear functional on $\ell^p$, 8 pt)**.** Let $\mathbb{K} = \mathbb{R}$, $p \in (1,\infty)$ and $y \in \ell^{\frac{p}{p-1}}$. Consider the map

$$T : \ell^p \to \mathbb{R}, \quad Tx := \lim_{n\to\infty} \sum_{i=1}^{n} x^i y^i.$$

(i) Show that $T$ is well-defined, i.e., the above limit exists.

(ii) Show that $T$ is linear.

(iii) Show that $T$ is bounded. Calculate its operator norm.

**Problem 3** (integral equation, 6 pt)**.** Let $y \in C([0,1],\mathbb{R})$ and $k \in C([0,1]^2,\mathbb{R})$ be such that

$$\|k\|_\infty < 1.$$

Show that there exists a solution $x \in C([0,1],\mathbb{R})$ of the equation

$$x(t) - \int_0^1 k(t,s)x(s)\,ds = y(t), \quad \forall t \in [0,1], \tag{1}$$

and that this solution is unique.

**Remark:** You may use the fact that for every $x \in C([0,1],\mathbb{R})$ the function $[0,1] \ni t \mapsto \int_0^1 k(t,s)x(s)\,ds \in \mathbb{R}$ is continuous.

**Problem 4** ($\ell^p$ is complete, 7 pt)**.** Let $p \in [1,\infty)$. Show that the norm $\|\cdot\|_p$ on $\ell^p$ is complete.

**Remark:** You do not need to show that $\ell^p$ is a vector space nor that $\|\cdot\|_p$ is a norm.

**Problem 5** (criterion for boundedness of an operator, 6 pt)**.** Let $H$ be a Hilbert space and $T : H \to H$ a linear map satisfying

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad \forall x, y \in H.$$

Prove that $T$ is bounded.

**Hint:** Use a result from the lecture.

**(More problems on the back.)**

**Problem 6** (criterion for boundedness of a set, 5 pt)**.** Let $\mathbb{K} = \mathbb{R}$, $X$ be a normed real vector space and $S \subseteq X$. Assume that for every bounded linear functional $x' : X \to \mathbb{R}$ the set $x'(S) \subseteq \mathbb{R}$ is bounded. Prove that $S$ is bounded, i.e., that

$$\sup_{x \in S} \|x\| < \infty.$$

**Hint:** Consider the canonical map $\iota_X$ and use one of the three main theorems about operators on Banach spaces.

**Problem 7** (dual space of $\ell^\infty/c_0$, 4 pt)**.** Let $\mathbb{K} = \mathbb{R}$. We denote

$$c_0 := \left\{ x \in \mathbb{R}^{\mathbb{N}} \,\middle|\, x^i \to 0, \text{ as } i \to \infty \right\}.$$

Prove that the dual space of $\ell^\infty/c_0$, equipped with the quotient norm, is nonzero.

**Remark:** You do *not* need to show that the quotient is a normed space.

**Problem 8** ($\ell^p$ reflexive, 6 pt)**.** Prove that for every $1 < p < \infty$ the space $\ell^p$ is reflexive.

**Hint:** Relate the canonical map $\iota_{\ell^p} : \ell^p \to (\ell^p)''$ to the map $\Phi_p : \ell^{p'} \to (\ell^p)'$.

**Problem 9** (spectrum of left-shift, 6 pt)**.** Find the spectrum of the left-shift operator

$$L : \ell^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}, \quad Lx := \big(x_2, x_3, \dots\big).$$