[MAX 40 points] For this problems there is no need to use large \((n > 3)\) matrix computations or programming. All derivations should be present in your answers.

1. The figure above, depicts the network of renaissance Florentine families, in which links represent business and marital ties. This network is difficult to cut into two connected components without removing as many as 4 edges, which, one may speculate, had something to do with
relative stability of the network in times of a political turmoil. Nevertheless, being locked in a struggle for political control of the city of Florence in 1430s, two factions eventually appeared dominant in this struggle: one revolved around the powerful Strozzis (A), and the other around the infamous Medicis (B). One may reconstruct these factions, as shown, by computing the sign of the elements of Fiedler’s vector shifted to feature 0 median.

(a) [1 pt] Calculate the empirical degree distribution \( p_k \), that is the fraction of all nodes with degree \( k = 1, 2, 3, \ldots \). Compute the cut quality of the indicated bisection.

(b) [4 pt] Show that the cut quality of the current partition cannot be improved by one family from Medicis switching sides, that is by moving one node B \( \rightarrow \) A.

(c) [10 pt] We will now consider a modern network from Sicily that has ties reaching out to a much larger number of nodes \( n \). The structure of the network is not revealed, but the frequencies of nodes with degree \( k = 1, 2, 3 \) are proportional to the ones in the Florentine network, and there are no nodes of degree larger than 3, i.e. \( p_k = 0 \) for \( k > 3 \). (N.B., as usual, there should hold \( \sum_{k>0} p_k = 1 \)).

Calculate the smallest fraction of edges, \( f \), that one has to remove uniformly at random to guarantee that the giant component does not exist. Compute the fraction of nodes in the largest connected component as a function of \( f \).
2. With some infections, infected individuals become susceptible upon recovery. Hence, we distinguish two compartments susceptible (S) and infected (I). Consider the following model:

\[ S + I \xrightarrow{\alpha} 2I \]

\[ I \xrightarrow{\beta} S \]

where \( \alpha, \beta > 0, \alpha \neq \beta \) are the rates. Let \( s(t) \) and \( x(t) \) denote concentration of correspondingly \( S \) and \( I \) species, with \( s(t) + x(t) = 1 \)

(a) [5 pt] Formulate the system of ordinary differential equations for \( s(t) \) and \( x(t) \) and show that this system can be well-represented by one differential equation for \( x(t) \), write down this equation.

(b) [10 pt] Find all fixed points of the form \((s^*, x^*)\), and classify their stability depending on the parameters, draw the bifurcation diagram (that is \( \lim_{t \to \infty} x(t) \) versus the parameter).

3. Three airports, Amsterdam, Milan and Paris are connected with multiple flights as shown. Each arch at the map denotes one flight connection. You can see that the number of flights is different between different cities. This is because the airport network was adapted to accommodate the uneven passenger flow. To estimate the number of customers, the hotel management applies the following Markovian model: every day a passenger \( X \) takes a different flight and stays in the city over night. This person cannot use other means of transport, so, for example, if \( X \) is in Paris, the next day \( X \) will be in Amsterdam or Milan. \( X \) is equally likely to take each flight.
(a) [3 pt] Write down the transition probability matrix for $X$.

(b) [3 pt] Suppose $X$ is in Paris, what is the probability $X$ will again be in Paris in two days?

(c) [4 pt] Given passenger $X$ is a representative model for all passengers, what fractions of overnight stays

$$0 \leq f_{Pa}, f_{Mi}, f_{Am} \leq 1, \quad f_{Pa} + f_{Mi} + f_{Am} = 1$$

do passengers spend at each city? Note that the symmetry implies that these values should be the same for Paris and Milan,

$$f_{Pa} = f_{Mi}.$$
Declaration

I hereby declare that I have made the answers for this exam myself, without the help of any other person or other aids than: the course literature / lecture notes / own notes.

Name          Signature

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