
Solution 1

(a) The union of $[0, 1]$ and $[2, 3)$ does not belong to $\mathcal{B}$, so $\mathcal{B}$ is not a topology. We will show it is a basis. First of all, if $x \in \mathbb{R}$ then $x \in [n, n+1)$ for an integer $n \in \mathbb{Z}$. Hence, $\mathbb{R} = \bigcup \mathcal{B}$.

Secondly, let $x \in [p_1, q_1) \cap [p_2, q_2)$, where $p_1, q_1, p_2, q_2$ are rational numbers such that $p_1 < q_1$ and $p_2 < q_2$. Then $p_2 \leq x < q_1$ and $p_1 \leq x < q_2$, hence $x \in \max(p_1, p_2), \min(q_1, q_2)) \subset [p_1, q_1) \cap [p_2, q_2)$. It follows that $\mathcal{B}$ is a basis.

(b) Since $\mathbb{Q}$ is countable, so is $\mathbb{Q} \times \mathbb{Q}$, and it follows that $\mathcal{B}$ is countable. It follows that $(\mathbb{R}, \mathcal{F})$ is second countable, hence also first countable.

(c) The intervals $(a, b)$ with $a, b \in \mathbb{R}$, $a < b$ form a basis $\mathcal{B}_e$ for the Euclidean topology on $\mathbb{R}$. If $x \in (a, b)$, there exist rational numbers $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, hence $x \in [p, q) \subset (a, b)$ and we see that $\mathcal{F}$ contains $\mathcal{B}_e$ and hence the Euclidean topology.

(d) Since $[0, 1) \in \mathcal{B}$ it follows that $[0, 1)$ is open for $\mathcal{F}$. Since $[0, 1]$ is closed for the Euclidean topology, it is so for $\mathcal{F}$. On the other hand, $\mathbb{R} \setminus [1, 2)$ is closed for $\mathcal{F}$, hence $[0, 1) = [0, 1] \setminus [1, 2)$ is closed for $\mathcal{F}$. From what we proved, $[0, 1)$ and $\mathbb{R} \setminus [0, 1)$ form a partition of $\mathbb{R}$ into sets from $\mathcal{F}$. Therefore, $(\mathbb{R}, \mathcal{F})$ is not connected.

(e) Define $U_n := [1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$ for $n \in \mathbb{Z}$, $n \geq 1$. The sets $U_n$ belong to $\mathcal{F}$ and have $[0, 1)$ as their union. Since the union is disjoint, the open cover $\{U_n\}_{n \geq 1}$ of $[0, 1)$ is infinite, and has no finite subcover. It follows that $[0, 1)$ is not compact. Now $[0, 1)$ is closed in $\mathbb{R}$ by (d), hence also closed in $[0, 1)$. Therefore, the latter set cannot be compact.

Solution 2

(a) The fibers of $\pi|_B : B \to \pi(B)$ form a partition of $B$. Clearly, $R_B$ is the equivalence relation determined by that partition. Thus, the equivalence classes of $R_B$ are the following fibers:

$$\pi|_B^{-1}(\pi(b)) = B \cap \pi^{-1}(\pi(b)) = B \cap R[b], \quad (b \in B).$$

These are the sets $\{(s, t)\}$ with $0 < s < 1$ and $|t| \leq 1$, and $\{(0, t), (1, -t)\}$ with $-1 \leq t \leq 1$.

(b) From the description of the equivalence classes we see that $X/R_B$ is the Möbius band.
(c) The equivalence classes for $R_B$ are the fibers of the map $g : B \to \pi(B)$ given by $g = \pi|_B$. It follows that there exists a map $\bar{g} : B/R_B \to \pi(B)$ such that $g \circ \pi_B = g$. Let $f$ be the composition of $\bar{g}$ with the inclusion map $i : \pi(B) \to X/R$ (or $\bar{g}$ viewed as map $B/R_B \to X/R$). Then $f \circ \pi_B = i \circ g = \pi_B$.

(d) If $\xi_1, \xi_2 \in B/R_B$ we write $\xi_j = \pi_B(b_j)$, for suitable $b_1, b_2 \in B$. From $f(\xi_1) = f(\xi_2)$ it follows that $\pi(b_j) = f(\pi_B(b_j)) = f(\xi_j)$ is independent of $j$. Therefore, $b_1 R_B b_2$ and we conclude $\xi_1 = \xi_2$. It follows that $f$ is injective.

Since $\pi$ is continuous $X \to X/R_B$, so is its restriction $\pi|_B$ and we see that $f \circ \pi_B : B \to X/R$ is continuous. It now follows from a proven property of the quotient topology on $B/R_B$ that $f : B/R_B \to X/R$ is continuous.

Finally, $B$ is (closed and bounded in $\mathbb{R}^2$ hence) compact and $\pi_B$ is continuous, hence $B/R_B$ is compact. Furthermore, $X/R$ is Hausdorff. We just proved that $f : B/R_B \to X/R$ is injective continuous. By a well known result it follows that $f$ is an embedding.

Solution 3

(a) Since $\varphi^2 = \text{id}_M$ it follows that $\varphi$ is bijective with inverse $\varphi^{-1} = \varphi$. It follows that both $\varphi$ and its inverse are continuous. Hence, $\varphi$ is a homeomorphism.

Clearly, $x R x$. If $x R y$, then either $y = x$ or $y = \varphi(x)$. In the latter case, $\varphi(y) = x$. Hence $x \in \{y, \varphi(y)\}$ and we see that $y R x$. Finally, if $x R y$ and $y R z$, then $y \in \{x, \varphi(x)\}$ and $z \in \{y, \varphi(y)\}$. If $y = x$ then $z \in \{x, \varphi(x)\}$. If $y = \varphi(x)$ then $z \in \{y, \varphi(y)\} = \{\varphi(x), x\}$. In both cases, $x R z$. If follows that $R$ is an equivalence relation.

Alternative: note that $\Gamma := \{\text{id}_M, \varphi\}$ with composition is a group of homeomorphisms, and $x R y \iff y \in \Gamma x$, so $R$ is an equivalence relation.

(b) Let $x \in \pi^{-1}(\pi(V))$. Then $x \in \pi(V)$ or we see that $y R x$. Finally, if $x \in \pi(V)$, then $x \in \{y, \varphi(y)\}$ and $z \in \{y, \varphi(y)\}$. Hence $x \in \pi(V)$. This shows that $\pi^{-1}(\pi(V)) \subseteq V \cup \varphi(V)$.

Conversely, if $x \in V \cup \varphi(V)$, then $x \in \pi(V) \cup \pi(\varphi(V)) = \pi(V)$. Hence the identity.

(c) If $U$ is open, then $\pi^{-1}(\pi(U))$ is open by (b), hence $\pi(U)$ is open for the quotient topology.

(d) The set $\varphi^{-1}(U_{j_m})$ is open, since $\varphi$ is continuous. Furthermore, this set contains $\varphi^{-1} \varphi(m) = m$. It follows that $V_m = U_{i_m} \cap \varphi^{-1}(U_{j_m})$ contains $m$, is open and satisfies the other properties.

(e) If $M$ is compact, then so is $\pi(M) = M/R$ by continuity of $M$. Conversely, assume that $M/R$ is compact. We will show that $M$ is compact. Let $\{U_i\}_{i \in I}$ be an open covering of $M$. Then there exist indices $i_n$ and $j_m$ with the properties of (d), since
Solution 4

(a) Clearly, $\psi f \in C(M)$ and $\text{supp}(\psi f) \subset \text{supp}\psi \cap \text{supp} f \subset U$. Since $\text{supp} f$ is compact and $\text{supp}\psi$ closed, it follows that $\text{supp}\psi \cap \text{supp} f$ is compact, hence $\psi f \in C_c(U)$.

(b) By (a) the map $I_\psi : C_c(M) \to \mathbb{R}$ is well-defined and linear. If $f \geq 0$, then $\psi f \geq 0$, so $I_\psi(f) = I(\psi f) \geq 0$, and we see that $I_\psi$ is a positive integral.

(c) Since $M$ is locally compact and second countable it is paracompact, hence allows partitions of unity. By the assumption, there exists an open cover $\{U_j\}_{j \in J}$ of $M$ and for each $j \in J$ a strictly positive integral on $U_j$. Let $\{\eta_j\}_{j \in J}$ be a partition of unity subordinate to $\{U_j\}_{j \in J}$. Then by (b), for each $j \in J$ the map $(I_j)_{\eta_j}$ is a positive integral on $M$. For each $f \in C_c(M)$ we have that only finitely many functions $\eta_j f$ are non-zero and have compact support contained in $U_j$, so

$$I(f) = \sum_{j \in J} I_j(\eta_j f) = \sum_{j \in J} (I_j)_{\eta_j}(f)$$

is a finite sum of positive real numbers. It readily follows that $I$ is a positive integral on $M$. If $I(f) = 0$ then each of the terms in the above sum must be zero, hence $\eta_j f = 0$ for all $j$. It follows that $f = \sum_{j \in J} \eta_j f = 0$. Therefore, $I$ is strictly positive.

(d) Since a topological manifold is locally compact Hausdorff and second countable, all of the above applies. Therefore, we just need to show that for each $m \in M$ there exists an open neighborhood $U$ and a positive integral $I$ on $U$. There exists an open neighborhood $U$ of $m$ which is homeomorphic to $\mathbb{R}^n$ which in turn is homeomorphic to $V := (0,1)^n$. Let $\chi : U \to V$ be a homeomorphism. The Riemann integral provides a strictly positive integral $I_r$ on $V$. For $f \in C_c(U)$ we note that $f \circ \chi^{-1} \in C_c(V)$ and we define $I(f) = I_r(f \circ \chi^{-1})$. Then $I$ is readily seen to be linear and positive. If $I(f) = 0$, then $f \circ \chi^{-1} = 0$ hence $f = 0$ on $U$ and since $\text{supp} f \subset U$ it follows that $f = 0$. Thus, $I$ is strictly positive.
Solution 5

(a) The function $\eta_i : X \to \mathbb{R}$ is continuous, and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$. Therefore, $V_i = \eta_i^{-1}(\mathbb{R} \setminus \{0\})$ is open in $X$. Let $x \in X$, then $\sum_{i \in I} \eta_i(x) = 1$ (with only finitely many $\eta_i$ different from zero). It follows that $\eta_i(x) \neq 0$ for at least one $i$, hence $x \in V_i$. We conclude that $X = \bigcup_{i \in I} V_i$, hence $\mathcal{V}$ is an open covering of $X$.

(b) By definition, $V_i = \text{supp} \eta_i$. Since $\{\eta_i\}$ is subordinate to $\mathcal{U}$, it follows that $V_i = \text{supp} \eta_i \subset U_i$.

(c) Since $V_i \subset \bigcap_i \subset U_i$, it follows that $\mathcal{V}$ is a refinement. It remains to be shown that $\mathcal{V}$ is locally finite. Let $x \in X$. Since the family $\{\text{supp} \eta_i\}_{i \in I}$ is locally finite, it follows that there exists a neighborhood $N$ of $x$ such that $I_N := \{i \in I \mid \text{supp} \eta_i \cap N \neq \emptyset\}$ is finite. If $V_i \cap N \neq \emptyset$, then $i \in I_N$, so the collection $\mathcal{V}_i$ is locally finite.

(d) First assume (1). Then by a theorem (2) is valid. Now assume (2). Then in the above we have shown that every open covering of $X$ has a locally finite refinement. By definition this implies (1).