Exercise 1. Consider
\[ X_1 = \{(x, y, z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = y = 0, z \geq 0)\}, \]
\[ X_2 = \{(x, y, z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = 0, y^2 + z^2 = 1, z \geq 0)\}, \]
\[ X_3 = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 = 1) \text{ or } (y = 0, z = 0, \frac{1}{2} < |x| < 1)\}, \]

(i) Show that \( X_1, X_2, X_3 \) are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures). (0.5 p)

(ii) Show that the one-point compactifications of \( X_1, X_2 \) and \( X_3 \) are homeomorphic to each other. (1 p)

Exercise 2. Given a polynomial \( p \in \mathbb{R}[X_0, X_1, \ldots, X_n] \), we denote by \( \mathcal{R}_p \) the set of reminders modulo \( p \). In other words,
\[ \mathcal{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p, \]
where \( R_p \) is the equivalence relation on \( \mathbb{R}[X_0, X_1, \ldots, X_n] \) given by
\[ R_p = \{(q_1, q_2) : \exists q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}. \]

We also denote by \( \pi_p : \mathbb{R}[X_0, X_1, \ldots, X_n] \rightarrow \mathcal{R}_p \) the resulting quotient map. Show that:

(i) There is a unique algebra structure on \( \mathcal{R}_p \) (i.e. unique operations +, \cdot and multiplications by scalars, defined on \( \mathcal{R}_p \)) with the property that \( \pi_p \) is an morphism of algebras, i.e.
\[ \pi_p(q_1 + q_2) = \pi_p(q_1) + \pi_p(q_2), \quad \pi_p(q_1 \cdot q_2) = \pi_p(q_1) \cdot \pi_p(q_2), \quad \lambda \pi_p(q) = \pi_p(\lambda q) \]
for all \( q_1, q_2 \in \mathbb{R}[X_0, X_1, \ldots, X_n], \lambda \in \mathbb{R}. \) (0.5 p)

(ii) For \( p = x_0^2 + \ldots + x_n^2 \), the spectrum of \( \mathcal{R}_p \) has only one point. (1 p)

(iii) For \( p = x_0^2 + \ldots + x_n^2 - 1 \), the spectrum of \( \mathcal{R}_p \) is homeomorphic to \( S^n \). (1 p)

(iv) What is the spectrum for \( p = x_0x_1 \ldots x_n \)? (0.5 p)

Exercise 3. Let \( X \) be the space of continuous maps \( f : [0, 1] \rightarrow [0, 1] \) with the property that \( f(0) = f(1) \). We endow it with the sup-metric \( d_{sup} \) and the induced topology (recall that \( d_{sup}(f, g) = sup\{|f(t) - g(t)| : t \in [0, 1]\} \)). Prove that:

(i) \( X \) is bounded and complete. (1 p)

(ii) \( X \) is not compact. (0.5 p)
**Exercise 4.** Show that:

(i) The product of two sequentially compact spaces is sequentially compact. (1 p)

(ii) Deduce that the product of two compact metric space is a compact space. (1 p)

**Exercise 5.** Show that the family of open intervals

\[ U := \{ (q, q + 1) : q \in \mathbb{R} \} \]

forms an open cover of \( \mathbb{R} \) which admits no finite sub-cover, but which admits a locally finite sub-cover. (1.5 p)

**Exercise 6.** Prove that there is no continuous injective map \( f : S^1 \lor S^1 \to S^1 \), where \( S^1 \lor S^1 \) is a bouquet of two circles (two copies of \( S^1 \), tangent to each other). (1.5 p)

Note: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 11 points in the game!).