Inleiding Topologie (January 29, 2014)

Note: The questions marked with "+" are worth 1 point, the rest 0.5 points. As aggreed, you are allowed to use during the exam the three sheets of A4 papers (= six pages) containing definitions, theorems, etc from the course- that you prepared at home. **For some explanations/hints, please see the end of the exam!!!!!!!!!!!!**.

Exercise 1. For $\mathbb{R}$ we consider the family $\mathcal{S}$ of subsets consisting of all the intervals of type (m, M) with $m < M < 0$, all intervals of type (m, M) with $0 < m < M$ (m and M real numbers) and the interval $[-1, 1)$. Denote by $\mathcal{T}$ the smallest topology on $\mathbb{R}$ containing $\mathcal{S}$.

a. show that $\mathcal{S}$ is not a topology basis and describe a basis of $(\mathbb{R}, \mathcal{T})$.

b. is $(\mathbb{R}, \mathcal{T})$ Hausdorff? Is it second countable?

c. find an interval of type $[a, b]$ whose closure inside $(\mathbb{R}, \mathcal{T})$ is not an interval.

d. find an interval of type (a, b) whose interior inside $(\mathbb{R}, \mathcal{T})$ is not an interval.

e. find an interval of type $[a, b]$ with the property that, together with the topology induced from $(\mathbb{R}, \mathcal{T})$, is not compact.

f. find an interval of type (a, b) with the property that, together with the topology induced from $(\mathbb{R}, \mathcal{T})$, is not connected.

g. (+) consider

$$f : (\mathbb{R}, \mathcal{T}) \longrightarrow (\mathbb{R}, \mathcal{T}_{\text{Eucl}}),\ f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } x \geq -1 \end{cases}$$

Is f continuous? Is f sequentially continuous?

Exercise 2. Let X be the open cylinder $(-1, 1) \times S^1$ and let Y be the open Moebius band (i.e. the Moebius band discussed in the lectures, from which the boundary circle was removed).

a. Describe the 1-point compactification X^+ as a subspace of $\mathbb{R}^3$.

b. (+) Describe $X^+ \subset \mathbb{R}^3$ by explicit formulas and write down an explicit embedding

$$f : X \longrightarrow \mathbb{R}^3$$

so that X^+ is the image of f together with the extra-point $(0, 0, 0)$.

c. Show that the 1-point compactification of Y is homeomorphic to the projective plane $\mathbb{P}^2$.

d. Show that X and Y are not homeomorphic.

Exercise 3. Let $X = [-1, 1] \times \mathbb{R}$.

a. (+) Find all the numbers $\lambda, a, b \in \mathbb{R}$ with the property that

$$n \cdot (x, y) := (\lambda^n x, a + by + \lambda n)$$

defines an action of the additive group $(\mathbb{Z}, +)$ on X.

b. (+) For which values of λ, a, b that you found is the resulting quotient X/Γ compact?

c. For λ, a, b from b., show that X/Γ is homeomorphic to the Moebius band.

Exercise 4. In this exercise we work over $\mathbb{R}$ (hence we consider real-valued functions and real algebras). Let X and Y be compact Hausdorff spaces. For $u \in \mathcal{C}(X)$, $v \in \mathcal{C}(Y)$, define $u \otimes v \in \mathcal{C}(X \times Y)$ given by

$$u \otimes v : X \times Y \longrightarrow \mathbb{R}, \quad (u \otimes v)(x, y) := u(x)v(y),$$

and we denote by $\mathcal{A} \subset \mathcal{C}(X \times Y)$ the set of functions of type

$$\sum_{i=1}^{k} u_i \otimes v_i \quad \text{with } k \in \mathbb{N}, u_1, \ldots, u_k \in \mathcal{C}(X), v_1, \ldots, v_k \in \mathcal{C}(Y)$$

Show that:

a. (+) $\mathcal{A}$ is a dense subalgebra of $\mathcal{C}(X \times Y)$.

b. For any $\chi \in X_{\mathcal{A}}$, $\chi_1 : \mathcal{C}(X) \longrightarrow \mathbb{R}$ and $\chi_2 : \mathcal{C}(Y) \longrightarrow \mathbb{R}$ given by

$$\chi_1(u) := \chi(u \otimes 1), \quad \chi_2(v) := \chi(1 \otimes v)$$

are characters.

c. (+) $X_{\mathcal{A}}$ is homeomorphic to $X \times Y$.

Notes/hints:

1. Please motivate all your answers. For instance, in Exercise 1, for b. do not just give an yes/no answsr, for, c.-f. prove why the intervals that you found do satisfy the required conditions, at point g. explain/prove why f is, or isn't, continuous or sequentially continuous, and similarly for the other exercises.

2. For items a. and c. of Exercise 2, and item c. of Exercise 3, you do not have to give explicit formulas; pictures are enough, provided they are properly explained.

3. Exercise 2: you may want to remember the models $T_{R,r}$ of the torus:

$$\begin{aligned} T_{R,r} &= \{(x, y, z \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\} = \\ &= \{(R + r\cos(a))\cos(b), (R + r\cos(a))\sin(b), r\sin(a)) : a, b \in [-\pi, \pi]\} \end{aligned}$$

(where, to obtain a torus, one has to assume $R > r > 0$). For d.: you may want to remember that $\mathbb{P}^2$ is a 2-dimensional topological manifold (in particular, each point has a neighborhood homeomorphic to $\mathbb{R}^2$).

4. In exercise 3, if you encounter 0^0, take it to be 1 (say by convention).