Exercise 1. (1p) Show that
\[ K := \{ (x, y) \in \mathbb{R}^2 : x^{2012} + y^{2012} \leq 10 \sin(e^x + e^y + 1000) + e^{\cos(x^2+y^2)} \} \]
is compact.

Exercise 2. (1.5 p) Let \( X \) be a bouquet of two circles:
\[ X = \{ (x, y) \in \mathbb{R}^2 : ((x − 1)^2 + y^2 − 1)((x + 1)^2 + y^2 − 1) = 0 \} \]
We say that a space \( Y \) is an exam-space if there exist three distinct points \( p, q, r \in X \) such that \( Y \) is homeomorphic to the one point compactification of \( X \setminus \{ p, q, r \} \).

Find the largest number \( l \) with the property that there exist exam-spaces \( Y_1, \ldots, Y_l \) with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 3. (1p) Let \( X \) be a topological space and let \( \gamma : [0, 1] \to X \) be a continuous function. Assume that \( \gamma \) is locally injective i.e., for any \( t \in [0, 1] \), there exists a neighborhood \( V \) of \( t \) in \( [0, 1] \) such that \( \gamma|_V : V \to X \) is injective. Show that, for any \( x \in X \), the set
\[ \gamma^{-1}(x) := \{ t \in [0, 1] : \gamma(t) = x \} \]
is finite.

Exercise 4. (1p) Let \( X \) be a normal space and let \( A \subset X \) be a subspace with the property that any two continuous functions \( f, g : X \to \mathbb{R} \) which coincide on \( A \) must coincide everywhere on \( X \). Show that \( A \) is dense in \( X \) (i.e. the closure of \( A \) in \( X \) coincides with \( X \)).

Exercise 5. (1p) Consider the following open cover of \( \mathbb{R} \):
\[ U := \{ (r, s) : r, s \in \mathbb{R}, |r − s| < \frac{1}{3} \} \]
Describe a locally finite subcover of \( U \).
Exercise 6. (each of the sub-questions is worth 0.5 p) Let $A$ be a commutative algebra over $\mathbb{R}$. Assume that it is finitely generated, i.e. there exist $a_1, \ldots, a_n \in A$ (called generators) such that any $a \in A$ can be written as

$$a = P(a_1, \ldots, a_n),$$

for some polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$. Recall that $X_A$ denotes the topological spectrum of $A$; consider the functions

$$f_i : X_A \longrightarrow \mathbb{R}, \quad f_i(\chi) = \chi(a_i) \quad 1 \leq i \leq n,$$

$$f = (f_1, \ldots, f_n) : X_A \longrightarrow \mathbb{R}^n.$$

Show that

(i) $f$ is continuous.

(ii) For any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$,

$$\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n)).$$

(iii) $f$ is injective.

(iv) the topology of $X_A$ is the smallest topology on $X_A$ with the property that all the functions $f_i$ are continuous.

(v) $f$ is an embedding.

Next, for a subspace $K \subset \mathbb{R}^n$, we denote by $\text{Pol}(K)$ the algebra of real-valued polynomial functions on $K$ and let $a_1, \ldots, a_n \in \text{Pol}(K)$ be given by

$$a_i : K \longrightarrow \mathbb{R}, \quad a_i(x_1, \ldots, x_n) = x_i.$$

Show that

(vi) $\text{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.

(vii) Show that the image of $f$ (from the previous part) contains $K$.

Finally:

(viii) For the $(n - 1)$ sphere $K = S^{n-1} \subset \mathbb{R}^n$, deduce that $f$ induces a homeomorphism between the spectrum of the algebra $\text{Pol}(K)$ and $K$.

(ix) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that the spectrum of $\text{Pol}(K)$ is homeomorphic to $K$?

Note: Motivate all your answers; give all details; please write clearly (English or Dutch). The mark is given by the formula:

$$\min\{10, 1 + p\},$$

where $p$ is the number of points you collect from the exercises.